Math2310 - Fall '22

Syllabus - Lecture 21

by Gennady Uraltsev

Topics

1 Surface integrals

1.1 Parameterized surfaces

- <u>`defn`</u> Parameterized surface
  - The parameterization $\Phi(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$
  - relation with the idea of "change of variables"
- <u>`exmpl`</u> Parameterizing the upper hemisphere inspired by cartesian coordinates:

$$\Phi(x,y) = \begin{pmatrix} x \\ y \\ \sqrt{1-x^2-y^2} \end{pmatrix} \qquad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x^2+y^2 \leq 1 \right\}$$

- <u>`exmpl`</u> Parameterizing the upper hemisphere inspired by cylindrical coordinates:

$$\Phi(\rho,\theta) = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \\ \sqrt{1-\rho^2} \end{pmatrix} \qquad \rho \in [0,1], \theta \in [0,2\pi]$$

- <u>`exmpl`</u> Parameterizing the upper hemisphere inspired by spherical coordinates:

$$\Phi(\phi,\theta) = \begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos\phi \end{pmatrix} \qquad \phi \in [0,\pi/2], \theta \in [0,2\pi].$$

- <u>`rmk`</u> the theorem of Invariance of domain - Wikipedia: to describe a parameterized surface using a smooth function you always need 2 variables:

1.2 Surface area surfaces

- Deducing the formula: the checkerboard grid obtained using the parameterization $\Phi : \Omega \to S \subset \mathbb{R}^3$ of a surface
- The geomtric meaning of the vectors $\partial_u \Phi(u,v)$ and $\partial_v \Phi(u,v)$

- The stretch factor of the squares of the grid:

$$\|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\|$$

- The order does not matter for the purposes of computing the area (it will matter for flows, later!)

- `exmpl` graphically identifying the vectors $\partial_\phi \Phi(\phi,\theta)$ and $\partial_\theta \Phi(\phi,\theta)$ for spherical coordinates:

$$\Phi(\phi,\theta) = \begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos\phi \end{pmatrix}$$

  spherical coordinates surface area - GeoGebra

- `exmpl` surface area of a sphere

- `exmpl` surface area of a spherical cap (imposing bounds on domain)

- surface area of a graph

  - recall the defintion of a graph of $f(x,y)$

  - the function $z = f(x,y)$ naturally induces a parameterization:

$$\Phi(u,v) = \begin{pmatrix} u \\ v \\ f(u,v) \end{pmatrix}$$

  - Computing the partials:

$$\partial_u \Phi(u,v) = \begin{pmatrix} 1 \\ 0 \\ \partial_u f(u,v) \end{pmatrix}, \qquad \partial_v \Phi(u,v) = \begin{pmatrix} 0 \\ 1 \\ \partial_v f(u,v) \end{pmatrix}.$$

  - Computing the dArea:

$$\|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\| = \left\| \begin{pmatrix} -\partial_u f(u,v) \\ -\partial_v f(u,v) \\ 1 \end{pmatrix} \right\|$$

  - the formula for the area of the graph:

$$\iint_{\mathcal{D}} \sqrt{1 + \partial_u f(u,v)^2 + \partial_v f(u,v)^2} \mathrm{d}u \mathrm{d}v$$

  - comparison with the 1D formula for the length of a curve described as a graph: $y = f(x)$:

$$\text{Lengt} = \int_{x=x_0}^{x_1} \sqrt{1 + f'(x)^2} \mathrm{d}x$$

  - geometric intuition and comparison

- `exmpl` surface area of a paraboloid over a disk

- `exmpl` surface area of a cone

References

Textbook

- [Ste] Chap 15.5 Surface area
- [Ste] Chap 15 complete **except** Chap 15.4 Applications of Double Integrals (skipped)

Videos

- The Surface Area formula for Parametric Surfaces // Vector Calculus - YouTube
- Introduction to the surface integral | Multivariable Calculus | Khan Academy - YouTube
- Parametrizing Surfaces, Surface Area, and Surface Integrals: Part 1 - YouTube

Geogebra applets

- spherical coordinates surface area - GeoGebra
- parameterized surfaces and dArea - GeoGebra