Math2310 - Fall '22

Syllabus - Lecture 20

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Review

- Basic integrals in 3D
- Fubini in 3D

Topics

1 Change of variable formula in 3D

- General change of variable formula
- The stretch factor of cubes: geometric intuition
- The determinant:
  - methods of computaiton (row or column exapansion)
  - meaning: the full dimensional volume (valid in 2D and 3D)
- The derminant of the Jacobian: comparison with the 2D case.
- `thm` 3D change of variable formula for integrals.

1.1 Imporant coordinate systems (changes of variable) in 3D:

- Important example: **cylindrical coordinates**
  - cylindrical coordinates are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta) \\ \rho\sin(\theta) \\ z \end{pmatrix} \qquad \theta \in [0, 2\pi], \rho \in [0, \infty), z \in \mathbb{R}$$

  - The Jacobian is ρ so you integrate w.r.t.

$$\rho \,\mathrm{d}\rho \,\mathrm{d}\theta \,\mathrm{d}z$$

- `exmpl` the volume of a solid of rotation (HW09 P2)

- Important example: **spherical coordinates**
    - Spherical coordinates are given by

$$\begin{pmatrix} x(\phi,\theta,\rho) \\ y(\phi,\theta,\rho) \\ z(\phi,\theta,\rho) \end{pmatrix} = \begin{pmatrix} \rho\sin\phi\cos(\theta) \\ \rho\sin\phi\sin(\theta) \\ \rho\cos\phi \end{pmatrix} \qquad \phi\in[0,\pi], \theta\in[0,2\pi], \rho\in[0,\infty)$$

    - The determinant of the Jacobian is $\rho^2\sin\phi$ so you integrate w.r.t.

$$\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z$$

    - Visualizing the vectors

$$\partial_\phi\begin{pmatrix} x(\phi,\theta,\rho) \\ y(\phi,\theta,\rho) \\ z(\phi,\theta,\rho) \end{pmatrix} \qquad \partial_\theta\begin{pmatrix} x(\phi,\theta,\rho) \\ y(\phi,\theta,\rho) \\ z(\phi,\theta,\rho) \end{pmatrix} \qquad \partial_\rho\begin{pmatrix} x(\phi,\theta,\rho) \\ y(\phi,\theta,\rho) \\ z(\phi,\theta,\rho) \end{pmatrix}$$

    - rmk The above vectors are orthogonal: geometric and alegbraic understanding
    - rmk the norms of the above vectors: geometric understanding and algebraic proof
    - Proof for the fact that the determinant of the Jacobian is $\rho^2\sin\phi$ using the above facts.

References

Textbook

- [Ste] Chap 15.7 (complete) - Triple Integrals in Cylindrical Coordinates
- [Ste] Chap 15.8 (complete) - Triple Integrals in Spherical Coordinates
- [Ste] Chap 15.9 (complete) - Change of variables in Multiple integrals
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Videos

- Calculus 3: Triple Integrals (3 of 25) Choosing a Coordinate System: Cylindrical - YouTube
- Integration in Spherical Coordinates - YouTube

Geogebra applets

- Cylindrical coordinates - GeoGebra
- spherical coordinates - GeoGebra