Math2310 - Fall '22

Syllabus - Lecture 18

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Review

1 Integrals over multivariable domains

- Fubini's theorem
- Integration techniques:
  - Describing regions and exchanging order of integration
  - Splitting regions
- changes of variables in 1D

Topics

1 Change of variable formula in 2D

1.1 Polar coordinates

- geometric intuition: polar change of variables - GeoGebra
- the strectch factor of the grid and dArea: $\rho \mathrm{d}\rho \mathrm{d}\theta$.
- integrating in polar coordinates
- `exmpl` integrating $f(x, y) = e^{-x^2 - y^2}$ over a disk
  - impossibility to do in cartesian coordinates
  - easy to do in polar coordinates
- other expamples

1.2 General change of variables in 2D

- General change of variable formula: motivation and example
- `defn` change of variable $\Phi : \mathcal{D} \subset \mathbb{R}^2 \to \tilde{\mathcal{D}} \subset \mathbb{R}^2$.
  - algebraic Φ example of $\Phi(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}$
  - graphical examples of $\Phi(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}$
  - imagining $\Phi(u, v)$: a derscription of deforming a grid

- changes of variables as strectching a piece of rubber
- properties of Φ: smoothness, injectivity (1-to-1), surjectivity (onto)

- ideas of proof and the determinant of the Jacobian as a "stretch factor"
  - The stretch factor of the small squares coming into the Riemann sum
  - geometric visualization of

$$\partial_u \Phi(u,v) \approx \Phi(u+\mathrm{d}u, v) - \Phi(u,v),$$
$$\partial_v \Phi(u,v) \approx \Phi(u, v+\mathrm{d}v) - \Phi(u,v).$$

  - geometric visualization of dArea under the action of Φ
  - the cross product to compute dArea:

$$\mathrm{dArea} = \left|\partial_u \Phi(u,v) \times \partial_v \Phi(u,v)\right| \mathrm{d}u\mathrm{d}v$$

  - the importance of the absolute value: comparison with 1D case. No negative areas!
- Computing the Jacobian for the polar change of variables.
- `exmpl` examples (general change of variables in 2D - example - GeoGebra):

$$f(x,y) = \frac{x}{y}$$

integrated over

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \in [1,2], 0 < y < \frac{1}{x} \right\}$$

  - computation using cartesian coordinates
  - the change of variable:

$$\begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} = \begin{pmatrix} u \\ v/u \end{pmatrix}$$

  - figuring out the bounds:

$$u \in [1,2]$$
$$v \in [0,1]$$

References

Textbook

- [Ste] Chap 15.1 (complete) - Double integrals over rectangles
- [Ste] Chap 15.2 (complete) - Double integrals general regions

Videos

- Defining Double Integration with Riemann Sums | Volume under a Surface - YouTube
- Converting double integrals to polar coordinates (KristaKingMath) - YouTube
- Double Integration Example over General Regions — two ways! - YouTube

Geogebra

- polar change of variables - GeoGebra
- general change of variables in 2D - example - GeoGebra