

# Time-Frequency Analysis of the Variational Carleson Operator using outer-measure $L^p$ spaces

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

GENNADY URALTSEV

aus

Leningrad, USSR

Bonn 2017

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Gutachter: Prof. Dr. Christoph Thiele
2. Gutachter: Prof. Dr. Massimiliano Gubinelli

Tag der Promotion: 14.07.2017  
Erscheinungsjahr: 2017

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# Summary

In this work we are concerned with developing a systematic framework for dealing with the Carleson operator

$$\mathcal{C}f(z) := \sup_c \left| \int_c^{+\infty} \widehat{f}(\xi) e^{2\pi i \xi z} d\xi \right| \quad (0.1)$$

and its variational counterpart given by

$$\mathcal{C}^r f(z) = \sup_{\dots < c_k < c_{k+1} < \dots} \left( \sum_{k \in \mathbb{Z}} \left| \int_{c_k}^{c_{k+1}} \widehat{f}(\xi) e^{2\pi i \xi z} \right|^r \right)^{1/r}. \quad (0.2)$$

The boundedness on  $L^p(\mathbb{R})$  with  $p \in (1, \infty)$  of these operators implies the famous Carleson's Theorem on the almost everywhere convergence of the Fourier integral for functions in  $L^p(\mathbb{R})$ . As a matter of fact, given  $f \in L^p(\mathbb{R})$  it holds that

$$\left\{ z \in \mathbb{R} : \lim_{N \rightarrow \infty} \int_{-N}^N \widehat{f}(\xi) e^{2\pi i \xi z} d\xi \text{ doesn't exist} \right\} \subset \{ z \in \mathbb{R} : \mathcal{C}^r f(z) = +\infty \}$$

for any  $r \in [1, \infty)$ . If  $\mathcal{C}^r f \in L^p(\mathbb{R})$  then the sets above have vanishing Lebesgue measure.

The Carleson operator is a prototypical operator with modulation symmetry and the main set of techniques for dealing with such operators is often referred to as time-frequency analysis. These techniques were originally introduced by Carleson in his seminal paper [Car66] on the convergence of Fourier series for  $L^2([-\pi/2, \pi/2])$  with many further advancements. Hunt [Hun68] extended Carleson's result to functions in  $L^p([-\pi/2, \pi/2])$  with  $p \in (2, \infty)$ . Fefferman [Fef73] gave an alternative proof of Carleson's result which actually introduced the operator (0.1). In [LT00] Lacey and Thiele gave another proof of boundedness of (0.1) that used elements of both Carleson's and Fefferman's approach in a setting that was generalized in [OSTTW12] to deal with (0.2). Time-frequency analysis techniques were also used to study other operators with modulation symmetries, like the Bilinear Hilbert Transform, that are beyond the scope of this exposition.

In this thesis we elaborate and expand on outer-measure  $L^p$  spaces introduced in [DT15] and therein applied to the Bilinear Hilbert Transform, an operator with the same symmetries as (0.1). The main novelty of this approach is that bounds are obtained for so-called embedding maps. Generally speaking, an embedding map provides a representation of a function by a set of coefficients on the symmetry space of the problem at hand. Outer-measure  $L^p$  spaces represent the correct functional framework for dealing with embedding maps. In turn, the bounds on the embedding maps allow one to bound operator at hand via a wave packet representation.

Furthermore, it has been shown in [DPDU16] that iterated outer-measure  $L^p$  spaces introduced in [Ura16] correctly encode the locality properties of the operators (0.1) and (0.2). In that paper, similarly to how it is done in [CDPO16] for the Bilinear Hilbert Transform, the authors manage

to deduce from the bounds on the embeddings that the Carleson and the Variational Carleson operators can be bounded by positive sparse forms. The theory of weighted bounds for sparse forms is well-understood (see [LN15] and [CDPO16]) and as a consequence we are able to give a complete answer to an open question about weighted bounds for operators (0.1) and (0.2). Partial progress for this problem has been made previously in [DL12a].

We now illustrate the main idea behind the reduction of the bounds for the operators (0.1) and (0.2) to the bounds for embedding maps. Suppose that  $f \in S(\mathbb{R})$  in the expressions (0.1) and (0.2). Fix a Borel-measurable stopping function  $c : \mathbb{R} \rightarrow \mathbb{R}$  and a function  $a \in S(\mathbb{R})$  or, in the case of (0.2), an increasing sequence  $\mathbf{c} : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R}$  of Borel-measurable stopping functions and a  $\mathbf{a} \in S(\mathbb{R}; l^{r'}(\mathbb{Z}))$  i.e. a smooth, rapidly decaying function with values in  $l^{r'}$  sequences. Then the wave packet representation

$$\int_{\mathbb{R}} \int_{c(z)}^{\infty} \widehat{f}(\xi) e^{2\pi i \xi z} a(z) d\xi dz = \int_{\mathbb{R}_+^3} F(y, \eta, t) A(y, \eta, t) dy d\eta dt \quad (0.3)$$

holds for the dual form of (0.1), while

$$\left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \int_{\mathbf{c}_k(z)}^{\mathbf{c}_{k+1}(z)} \widehat{f}(\xi) e^{2\pi i \xi z} \mathbf{a}_k(z) d\xi dz \right| \leq \int_{\mathbb{R}_+^3} F(y, \eta, t) \mathfrak{A}(y, \eta, t) dy d\eta dt \quad (0.4)$$

holds for the dual form of (0.2).

The space  $\mathbb{R}_+^3 := \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$  appearing on the right parametrizes the translation, modulation (Fourier translation), and dilation symmetries. The embedded function  $F$  is given by

$$F(y, \eta, t) := f * \psi_{\eta, t}(y) \quad \text{with } \psi_{\eta, t}(y) := e^{i\eta y} t^{-1} \psi\left(\frac{y}{t}\right) \quad (0.5)$$

where  $\psi \in S(\mathbb{R})$  is an appropriately chosen base wavelet. The embedded functions  $A$  and  $\mathfrak{A}$  are defined respectively as

$$\begin{aligned} A(y, \eta, t) &:= \int_{\mathbb{R}} a(z) \psi_{\eta, t}^{c(z)}(y - z) dz \\ \mathfrak{A}(y, \eta, t) &:= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \mathbf{a}_k(z) \psi_{\eta, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(y - z) dz \end{aligned} \quad (0.6)$$

where  $\psi_{\eta, t}^c$  and  $\psi_{\eta, t}^{c_-, c_+}$  are “truncated” wave packets whose definition is somewhat more involved and can be found in Chapter 2.

The main result of [Ura16] (Chapter 2) consists of showing that bounds

$$\|A\|_{L^p \mathbb{E}^q S^m} \lesssim_{p, q} \|a\|_{L^p} \quad \forall p \in (1, \infty], q \in (1, \infty] \quad (0.7)$$

$$\|\mathfrak{A}\|_{L^p \mathbb{E}^q S^m} \lesssim_{p, q} \|\mathbf{a}\|_{L^p l^{r'}} \quad \forall p \in (1, \infty], q \in (r', \infty], r' \in [1, 2) \quad (0.8)$$

$$\|F\|_{L^p \mathbb{E}^q S^e} \lesssim_{p, q} \|f\|_{L^p} \quad \forall p \in (1, \infty], q \in (\min(2, p'), \infty] \quad (0.9)$$

hold with a constant independent of the stopping functions  $c$  and  $\mathbf{c}$  appearing in (0.6). The quasi-norms appearing on the left are a shorthand notation for the so called iterated outer-measure  $L^p$  quasi-norms. This is the main novelty of the approach of [Ura16] with respect to previous works that use the outer-measure  $L^p$  space framework ([DT15], [DPO15]).

Abstract results about outer-measure spaces imply that

$$\left| \int_{\mathbb{R}_+^3} F(y, \eta, t) A(y, \eta, t) dy d\eta dt \right| \lesssim \|F\|_{L^p \mathbb{E}^q S^e} \|A\|_{L^p l^{q'} S^m}$$

and

$$\left| \int_{\mathbb{R}_+^3} F(y, \eta, t) \mathfrak{A}(y, \eta, t) dy d\eta dt \right| \lesssim \|F\|_{L^p \mathbb{E}^q S^e} \|\mathfrak{A}\|_{L^{p'} \mathbb{E}^{q'} S^m}$$

as long as  $(p, p')$  and  $(q, q')$  are Hölder dual exponents i.e.  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{q} + \frac{1}{q'} = 1$ . The representation (0.3) with bounds (0.7) and (0.9) thus imply the bounds

$$\left\| \mathcal{C}f \right\|_{L^p} \lesssim_p \|f\|_{L^p} \quad \forall p \in (1, \infty).$$

Similarly, the representation (0.4) and bounds (0.8) and (0.9) imply that the bounds

$$\left\| \mathcal{C}^r f \right\|_{L^p} \lesssim_p \|f\|_{L^p} \quad \forall p \in (r', \infty)$$

hold as long as  $r \in (2, \infty]$ . This is known from [OSTTW12] to be the complete range of exponents where strong bounds for (0.2) hold.

Furthermore, bounds by sparse forms can be obtained from the wave packet representations (0.3) and (0.4) and bounds (0.7), (0.8), and (0.9). A sparse bilinear form is a map of the form

$$(f, a) \mapsto \sum_{I \in \mathfrak{S}} |I| \left( \int_I |f|^s \right)^{1/s} \left( \int_I |a|^t \right)^{1/t}$$

for  $s, t \in (1, \infty)$ , where  $\mathfrak{S}$  is a sparse grid. We say that a collection of intervals  $\mathfrak{S}$  is a sparse grid if there exists a constant  $C > 1$  such that for any interval  $I$  it holds that

$$\sum_{\substack{J \in \mathfrak{S} \\ J \subset I}} |J| \leq C|I|. \quad (0.10)$$

If  $p > s$ ,  $p' > t$  and  $\frac{1}{p} + \frac{1}{p'} = 1$ , we show that sparse forms associated with sparse grids with uniform sparseness constants are bounded uniformly on  $L^p \times L^{p'}$ . Furthermore, weighted  $L^p$  theory is also well-established for such forms, but it is beyond the scope of this exposition. It can be shown that the dual forms to (0.1) and (0.2) can be bounded by sparse forms in the sense that

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{C}f(x)a(x)dx \right| &\lesssim_s \sup_{\mathfrak{S}} \sum_{I \in \mathfrak{S}} |I| \left( \int_I |f|^s \right)^{1/s} \int_I |a| \quad \forall s > 1 \\ \left| \int_{\mathbb{R}} \mathcal{C}^r f(x)a(x)dx \right| &\lesssim_{r,s} \sup_{\mathfrak{S}} \sum_{I \in \mathfrak{S}} |I| \left( \int_I |f|^s \right)^{1/s} \int_I |a| \quad \forall s > r' \end{aligned} \quad (0.11)$$

where the supremum is taken over all sparse grids  $\mathfrak{S}$  with uniform sparseness.

## Outline

This thesis is structured into three Chapters. Chapter 1 contains an introduction to outer measure spaces and an outline of the proof of the bounds of the Carleson Operator (0.1) in the simplified Walsh case.

In Section 1.1 we present the generalities of outer-measure spaces. We begin with the definitions and some basic examples; then we continue with the most important properties of outer-measure spaces such as the outer-measure Hölder's inequality, interpolation properties, and domination results. This section generally follows [DT15], albeit with a somewhat different notation. Some proofs are omitted and can be found in the above paper.

Next, in Section 1.2 we present some basic results from time-scale analysis i.e. Calderón-Zygmund theory. We do not claim originality, generality, nor completeness but rather we aim at showing that outer-measure  $L^p$  framework has enough flexibility to express classical concepts. The full power of outer-measure  $L^p$  space approach can be seen in the following sections related to time-frequency analysis.

Sections 1.3 and 1.4 are respectively dedicated to presenting the results of [Ura16] and [DPDU16] for the technically simpler Walsh model of (0.1). We also avoid dealing with the variational counterpart (0.2).

In Section 1.3.1 we begin by introducing the Walsh group as an effective model for dealing with translation, modulation, and dilation symmetries. In Section 1.3.2 we prove the wave packet representation (0.3) for the real version of the Carleson operator. In Section 1.3.3 we use this representation to correctly deduce the Walsh model for the Carleson operator and to introduce the outer-measure  $L^p$  space structure on the Walsh model for the time-frequency plane. We also reduce the bounds for the Walsh Carleson operator to the bounds on the Walsh analogues of the embedding maps (0.5) and (0.6). In Sections 1.3.4 and 1.3.5 we prove the bounds on these embedding maps.

In Section 1.4 we talk about iterated outer-measure  $L^p$  spaces the in the Walsh model case. In Section 1.4.1 we prove the bounds (0.9) in the Walsh model case. In Section 1.4.2 we prove the bounds (0.7) in the Walsh model case. Finally in Section 1.4.3 we show how these bounds can be used to obtain sparse domination for the Walsh model of (0.1).

Chapter 2 contains the results of the paper [Ura16] while Chapter 3 contains the results of the paper [DPDU16].



# Chapter 1

## The Carleson Operator and outer measure spaces

Where is horse?

–S.T.

### 1.1 Generalities of outer measure spaces

Outer-measure  $L^p$  space requires introducing two objects: an *outer measure* that measures “how large” a set is and a *size* that measures “how large” functions are.

We will restrict to working with a space  $\mathbb{X}$  that is a separable complete metric space (Polish space). While the theory can possibly be developed in greater generality, this is beyond the scope of this exposition.

#### 1.1.1 Outer measures and sizes

The concept of an outer measure is, by itself, classical. Standard construction of the Lebesgue measure theory makes an interim use of an outer measure and then restricts to considering Carathéodory measurable sets. In the following, we do not restrict to such a “good” class of sets: in many applications this class would be trivial.

**Definition 1.1** (Outer measure). *An outer measure  $\mu$  on  $\mathbb{X}$  is a positive, monotone,  $\sigma$ -sub-additive set function i.e. a function  $\mu : \mathcal{P}(\mathbb{X}) \rightarrow [0, +\infty]$  defined on subsets of  $\mathbb{X}$  that satisfies the following properties:*

1.  $\mu(\emptyset) = 0$ ;
2. (monotonicity) given two subsets  $E, E' \subset \mathbb{X}$

$$E \subset E' \implies \mu(E) \leq \mu(E');$$

3. ( $\sigma$ -subadditivity) for any countable collection  $(E_n \subset \mathbb{X})_{n \in \mathbb{N}}$  of subsets of  $\mathbb{X}$  one has

$$\mu\left(\bigcup_{n \in \mathbb{N}} E_n\right) \leq \sum_{n \in \mathbb{N}} \mu(E_n).$$

We call the pair  $(\mathbb{X}, \mu)$  an outer measure space. For example,  $\mathbb{R}^n$  endowed with the Carathéodory outer measure associated to the Lebesgue measure is an outer measure space.

Outer measures are often may be generated by a pre-measure (term used here with a somewhat different meaning than in literature). Let us fix a distinguished collection  $\mathbb{T}$  of Borel subsets of  $\mathbb{X}$  and a set function  $\bar{\mu} : \mathbb{T} \rightarrow [0, +\infty]$ . The outer measure generated by  $\bar{\mu}$ , defined on any subset  $E \subset \mathbb{X}$  is given by

$$\mu(E) = \inf \left\{ \sum_{T \in \mathcal{T}} \bar{\mu}(T) : \bigcup_{T \in \mathcal{T}} T \supset E \right\}. \quad (1.1)$$

The lower bound is taken over all *countable* coverings  $\mathcal{T}$  of  $E$  made of generating sets from  $\mathbb{T}$ . Clearly,  $\mu$  is an outer measure. While it is often the case, it is not generally true that  $\mu(T) = \bar{\mu}(T)$  for  $T \in \mathbb{T}$ .

We say the  $\mathbb{X}$  is  $\sigma$ -finite with respect to the outer measure  $\mu$  if

$$\mathbb{X} = \bigcup_{n \in \mathbb{N}} \mathbb{X}_n \quad \mu(\mathbb{X}_n) < \infty. \quad (1.2)$$

If  $\bar{\mu} < \infty$  on  $\mathbb{T}$  and  $\mathbb{X} = \bigcup_{T \in \mathbb{T}} T$  then  $(\mathbb{X}, \mu)$  is  $\sigma$ -finite where  $\mu$  is generated by  $\bar{\mu}$ .

Apart from the classical Lebesgue-Carathéodory outer measure, another important example of an outer measure space is given by the upper half-space

$$\mathbb{R}_+^2 = \left\{ (x, s) \in \mathbb{R} \times \mathbb{R}^+ \right\}$$

endowed with Carleson tents (see figure 1.1) as the collection  $\mathbb{T} = \{T\}$  of generating sets:

$$T(x, s) = \{(y, t) \in \mathbb{R}_+^2 : |y - x| < s - t, t < s\}.$$

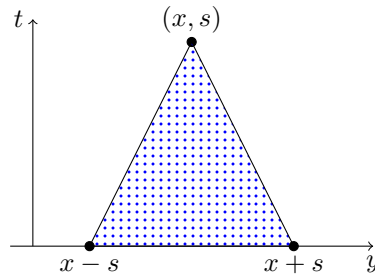


Figure 1.1: The tent  $T(x, s)$ .

We endow these sets with the pre-measure

$$\bar{\mu}(T(x, s)) = s.$$

Geometrically, if we associate to each point  $(x, s) \in \mathbb{R}_+^2$  the ball  $B_s(x) = (x - s, x + s)$  of the real line, then the tent  $T(x, s)$  is the set of all balls  $B_t(y)$  contained in  $B_s(x)$  while  $\mu(T(x, s)) = s = \frac{|B_s(x)|}{2}$

Recall that a set  $E$  is Carathéodory measurable if “it can be used to cut arbitrary sets” i.e. for any  $A \subset \mathbb{R} \times \mathbb{R}^+$  it must hold that

$$\mu(A) = \mu(A \cap E) + \mu(A \cap E^c).$$

We now show that the only Carathéodory measurable sets of this space are  $\emptyset$  and  $\mathbb{X}$  itself. Suppose  $E \subset \mathbb{R} \times \mathbb{R}^+$  has a non-empty boundary so that we may choose  $(x_0, s_0) \in \partial E$ . For an arbitrarily small  $\varepsilon > 0$  let  $A = T(x_0, s + \varepsilon)$  and since  $(x_0, s_0) \in \partial E$  there exists points  $(x', s') \in A \cap E$  and  $(x'', s'') \in A \cap E^c$  with  $s - \varepsilon < s', s'' < s + \varepsilon$ . It can be shown that  $\mu(A \cap E) \geq s' > s - \varepsilon$  since  $(x', s') \in T(y, t)$  only if  $t > s$  and similarly  $\mu(A \cap E^c) > s - \varepsilon$ . Then  $E$  is measurable only if

$$2s - 2\varepsilon < \mu(A \cap E) + \mu(A \cap E^c) = \mu(A) \leq s + \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary this leads to a contradiction.

We now introduce the concept of a *size*. It essentially is a quasi-norm on Borel functions that is lower semi-continuous with respect to pointwise convergence.

**Definition 1.2 (Size).** *A size  $\|\cdot\|_S$  is a functional on the set of all Borel measurable functions on  $\mathbb{X}$ , with values in  $[0, \infty]$ , that satisfies:*

1. (vanishing) if  $F = 0$  then  $\|F\|_S = 0$ ;
2. (homogeneity) for any  $\lambda \in \mathbb{C}$  it holds that  $\|\lambda F\|_S = |\lambda| \|F\|_S$  ;
3. (quasi-monotonicity) for any two Borel functions  $F$  and  $G$  one has

$$|F| < |G| \implies \|F\|_S \lesssim \|G\|_S;$$

4. ( $\sigma$ -quasi-triangle inequality) there exists a triangle constant  $c_s$  such that for any sequence of Borel functions  $F_n$  one has

$$\left\| \sum_{n \in \mathbb{N}} F_n \right\|_S \leq \sum_{n \in \mathbb{N}} c_s^{n+1} \|F_n\|_S \quad (1.3)$$

as long as the sum on the left converges pointwise.

Similarly to pre-measures, sizes may be generated by “local” sizes. Consider a distinguished collection  $\mathbb{T}$  of subsets of  $\mathbb{X}$  and suppose that they are Borel measurable. For each  $T \in \mathbb{T}$  let there be a size  $\|\cdot\|_{S(T)}$  defined on Borel functions on  $T$ . We say that  $\|\cdot\|_{S(T)}$  generate  $\|\cdot\|_S$  if

$$\|F\|_S = \sup_{T \in \mathbb{T}} \|F\|_{S(T)} \quad (1.4)$$

### 1.1.2 Outer $L^p$ spaces

Given the an outer measure and a size we can introduce an outer-measure integral and outer-measure  $L^p$  spaces.

The outer- $L^p$  quasi-norms for  $p \in (0, \infty)$  are give by

$$\|G\|_{L^p S}^p := \int_{\lambda \in \mathbb{R}^+} p \lambda^p \mu(\|G\|_S > \lambda) \frac{d\lambda}{\lambda}; \quad (1.5)$$

weak outer  $L^p$  quasi-norms are similarly given by

$$\|G\|_{L^{p, \infty} S}^p := \sup_{\lambda \in \mathbb{R}^+} \lambda^p \mu(\|G\|_S > \lambda). \quad (1.6)$$

The  $S, \mu$  - super-level outer measure is given by

$$\mu(\|G\|_S > \lambda) := \inf \{ \mu(E_\lambda) : \|G \mathbf{1}_{\mathbb{X} \setminus E_\lambda}\|_S \leq \lambda \} \quad (1.7)$$

where the lower bound is taken over Borel subset  $E_\lambda$  of  $\mathbb{X}$ .

The outer  $L^p$  spaces are subspaces of Borel functions on  $\mathbb{X}$  for which the above norms are finite. The expressions defining outer  $L^p$  quasi-norms are based on the super-level set representation of the Lebesgue integral, however the expression  $\mu(\|G\|_S > \lambda)$  that appears in lieu of the classical  $\mu(\{x : |g(x)| > \lambda\})$  if generally not a measure of any specific set.

Intuitively, outer measure  $L^p$  spaces represent interpolation spaces between the outer measure of the support of a function and the size. Given a Borel function  $F$  we say that  $\text{spt}(G) \subset E$  if  $\|\mathbb{1}_{\mathbb{X} \setminus E} G\|_S = 0$  and we define

$$\mu(\text{spt}(G)) = \inf \left\{ \mu(E) : \text{spt}(G) \subset E \right\}.$$

Given two functions  $G_1$  and  $G_2$  we identify them if  $\mu(\text{spt}(G_1 - G_2)) = 0$ . From now on we denote by  $G \in \mathcal{B}(\mathbb{X})$  the equivalence classes of Borel functions w.r.t to this relation. We also introduce the convention that  $+\infty \cdot 0 = 0 \cdot +\infty = 0$ .

Let us make two examples of outer measure spaces and sizes. First of all, outer measure  $L^p$  spaces encompass Lebesgue spaces. As a matter of fact consider  $\mathbb{R}^n$  and let the generating collection  $\mathbb{T}$  consist of balls  $B_r(x) = \{y : |y - x| < r\}$  with rational centers  $x \in \mathbb{Q}^d$  and rational radii  $r \in \mathbb{Q}^+$ . Let

$$\bar{\mu}(B_r(x)) = |B_r(x)| \approx_n r^n$$

so that the generated outer measure  $\mu$  becomes the familiar Carathéodory outer measure obtained via countable coverings with Euclidean balls. Set the size  $\|\cdot\|_S$  to be

$$\|f\|_S = \sup_{x \in \mathbb{R}^n} |f(x)|.$$

so that it clearly satisfies all the conditions. Notice that

$$\mu(\|f\|_S > \lambda) = \mathcal{L}^n(\{x : |f(x)| > \lambda\})$$

where  $\mathcal{L}^n$  is the standard Lebesgue measure on  $\mathbb{R}^n$ . This follows since if  $\|f \mathbb{1}_{\mathbb{R}^n \setminus E}\|_S \leq \lambda$  then  $E \supset \{x : |f(x)| > \lambda\}$  and on Borel sets one has  $\mu = \mathcal{L}^n$ .

The size  $S$  can be generated by a family of local sizes. For every ball  $B \in \mathbb{T}$  define the size  $\|\cdot\|_{S(B)}$  as

$$\|f\|_{S(B)} := \int_B |f(x)| dx = \frac{1}{|B|} \int_B |f(x)| dx.$$

By the Lebesgue differentiation theorem

$$\|f\|_S = \sup_B \|f\|_{S(B)}.$$

Thus the integral defined by (1.5) coincides with the classical Lebesgue definition. While in this example using integral type local sizes is a meaningless complication, this approach presents significant advantages for the applications in this thesis.

### 1.1.3 Properties of outer measure $L^p$ spaces

Outer measure  $L^p$  spaces have many important properties related to interpolation. We begin by illustrating quasi-subadditivity, Chebyshev's inequality, logarithmic convexity of outer  $L^p$  norms, the outer Hölder inequalities, and real interpolation. We will also illustrate a useful measure atomic decomposition property. Some of the more straightforward proofs will be omitted.

**Proposition 1.3** (Quasi subadditivity). *For any  $p \in (0, \infty)$  we have that  $\|F + G\|_{L^p S} \lesssim_p \|F\|_{L^p S} + \|G\|_{L^p S}$*

**Proposition 1.4** (Chebyshev's inequality). *For any  $p \in (0, \infty)$  we have that  $\|F\|_{L^p, \infty S} \lesssim_p \|F\|_{L^p S}$  i.e. for any  $\lambda > 0$*

$$\mu(\|F\mathbb{1}_{\mathbb{X} \setminus E_\lambda}\|_S > \lambda) \lesssim_p \frac{\|F\|_{L^p S}^p}{\lambda^p}$$

**Proposition 1.5** (Logarithmic convexity of  $L^p$  norms). *Let  $(\mathbb{X}, \mu)$  be an outer measure space with size  $\|\cdot\|_S$  and let  $F$  be a Borel function on  $\mathbb{X}$ . For every  $\theta \in (0, 1)$  and for  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  with  $p_0, p_1 \in (0, \infty]$ ,  $p_0 \neq p_1$  the inequality*

$$\|F\|_{L^{p_\theta} S} \leq C_{\theta, p_0, p_1} \|F\|_{L^{p_0, \infty} S}^{1-\theta} \|F\|_{L^{p_1, \infty} S}^\theta$$

holds.

*Proof.* Suppose  $p_0 < p_1$  and apply the definition (1.5) to obtain

$$\begin{aligned} \|F\|_{L^{p_\theta} S}^{p_\theta} &= \int_{\mathbb{R}^+} p_\theta \lambda^{p_\theta} \mu(\|F\|_S > \lambda) \frac{d\lambda}{\lambda} \\ &\leq p_\theta \|F\|_{L^{p_0, \infty} S}^{p_0} \int_0^\tau \lambda^{p_\theta - p_0} \frac{d\lambda}{\lambda} + p_\theta \|F\|_{L^{p_1, \infty} S}^{p_1} \int_\tau^\infty \lambda^{p_\theta - p_1} \frac{d\lambda}{\lambda} \\ &= p_\theta (p_\theta - p_0) \|F\|_{L^{p_1, \infty} S}^{p_0} \tau^{p_\theta - p_0} + p_\theta (p_1 - p_\theta) \|F\|_{L^{p_1, \infty} S}^{p_1} \tau^{p_\theta - p_1} \end{aligned}$$

for any  $\tau > 0$ . Optimizing in  $\tau$  gives the result.  $\square$

**Proposition 1.6** (Outer Hölder inequality). *Let  $(\mathbb{X}, \mu)$  be an outer measure space endowed with three sizes  $\|\cdot\|_S$ ,  $\|\cdot\|_{S'}$ , and  $\|\cdot\|_{S''}$  such that for any Borel functions  $F$  and  $G$  on  $\mathbb{X}$  the product estimate for sizes*

$$\|FG\|_S \lesssim \|F\|_{S'} \|G\|_{S''} \quad (1.8)$$

holds. Then for any Borel functions  $F$  and  $G$  on  $\mathbb{X}$  the following outer Hölder inequality holds:

$$\|FG\|_{L^p S} \lesssim \|F\|_{L^{p'} S'} \|G\|_{L^{p''} S''} \quad (1.9)$$

for any triple  $p, p', p'' \in (0, \infty]$  of exponents such that  $\frac{1}{p'} + \frac{1}{p''} = \frac{1}{p}$ .

Notice however that we do not claim that Hölder's inequality for outer measure  $L^p$  spaces holds with constant 1, even if the product estimate for sizes does.

*Proof.* Suppose by homogeneity that  $\|F\|_{L^{p'} S'} = \|G\|_{L^{p''} S''} = 1$ . Recall that

$$\|FG\|_{L^p S}^p = p \int_0^\infty \lambda^p \mu(\|FG\|_S > \lambda) \frac{d\lambda}{\lambda}.$$

The crucial observation is that for some  $C > 0$

$$\mu(\|FG\|_S > \lambda) \leq \mu(\|F\|_{S'} > C^{-1} \lambda^{p/p'}) + \mu(\|G\|_{S''} > C^{-1} \lambda^{p/p''}). \quad (1.10)$$

As a matter of fact let  $V_\lambda, W_\lambda \subset \mathbb{X}$  be two subsets such that

$$\begin{aligned} \mu(V_\lambda) &< 2\mu(\|F\|_{S'} > C^{-1} \lambda^{p/p'}) & \|F\mathbb{1}_{\mathbb{X} \setminus V_\lambda}\|_{S'} &\leq C^{-1} \lambda^{p/p'} \\ \mu(W_\lambda) &< 2\mu(\|G\|_{S''} > C^{-1} \lambda^{p/p''}) & \|G\mathbb{1}_{\mathbb{X} \setminus W_\lambda}\|_{S''} &\leq C^{-1} \lambda^{p/p''} \end{aligned}$$

so that

$$\|FG \mathbb{1}_{\mathbb{X} \setminus (V_\lambda \cup W_\lambda)}\|_S \lesssim C^{-2} \lambda^{p/p'} \lambda^{p/p''}$$

follows from (1.8) and (1.10) holds as long as  $C$  is chosen large enough. We use (1.10) and a change of variable in  $\lambda$  to write

$$\begin{aligned} \|FG\|_{L^p S}^p &\lesssim \int_0^\infty \lambda^p \mu(\|F\| > \lambda^{p/p'}) \frac{d\lambda}{\lambda} + \int_0^\infty \lambda^p \mu(\|G\| > \lambda^{p/p''}) \frac{d\lambda}{\lambda} \\ &\lesssim \int_0^\infty \lambda^{p'} \mu(\|F\| > \lambda) \frac{d\lambda}{\lambda} + \int_0^\infty \lambda^{p''} \mu(\|G\| > \lambda) \frac{d\lambda}{\lambda} \lesssim 1 \end{aligned}$$

which concludes the proof.  $\square$

**Proposition 1.7** (Marcinkiewicz interpolation). *Let  $(\mathbb{X}, \mu)$  and  $(\mathbb{Y}, \nu)$  be two outer measure space with sizes  $\|\cdot\|_{S_\mathbb{X}}$  and  $\|\cdot\|_{S_\mathbb{Y}}$  respectively. Assume  $p_0, p_1, q_0, q_1 \in (0, \infty]$  and let  $T$  be an operator that maps  $L^{p_0} S_\mathbb{X} + L^{p_1} S_\mathbb{X}$  to Borel function on  $\mathbb{X}$  such that it satisfies*

**scaling**  $|T(\lambda F)| = |\lambda T(F)|$  for all  $F \in L^{p_0} S_\mathbb{X} + L^{p_1} S_\mathbb{X}$  and  $\lambda \in \mathbb{C}$ ;

**quasi sub-additivity**

$|T(F+G)| \leq C(|T(F)| + |T(G)|)$  for all  $F, G \in L^{p_0} S_\mathbb{X} + L^{p_1} S_\mathbb{X}$  and some  $C \geq 1$ ;

**weak boundedness**

$$\begin{aligned} \|T(F)\|_{L^{q_0, \infty} S_\mathbb{Y}} &\leq C_0 \|F\|_{L^{p_0} S_\mathbb{X}} \quad \forall F \in L^{p_0} S_\mathbb{X} \\ \|T(F)\|_{L^{q_1, \infty} S_\mathbb{Y}} &\leq C_1 \|F\|_{L^{p_1} S_\mathbb{X}} \quad \forall F \in L^{p_1} S_\mathbb{X} \end{aligned}$$

Then for any  $\theta \in (0, 1)$ ,  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ , and  $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$  it holds that

$$\|T(f)\|_{L^{q_\theta} S_\mathbb{Y}} \lesssim_{\theta, p_0, 1, q_0, 1} C_0^{1-\theta} C_1^\theta \|F\|_{L^{p_\theta} S_\mathbb{X}}$$

We omit the proof of the above Proposition. It follows along the lines of classical real interpolation results. A proof can be found in [DT15].

We finally pass to an important atomic decomposition property for outer  $L^p$  spaces. A  $p$ -atom is a function  $F \in \mathcal{B}(\mathbb{X})$  such that

$$\mu(\text{spt}(F))^{1/p} \|F\|_S = 1 \tag{1.11}$$

**Proposition 1.8** ( $p$ -atomic decomposition). *If  $\|F\|_{L^p S} < \infty$  there exists a decomposition*

$$F = \sum_{k \in \mathbb{Z}} \lambda_k F_k \quad \|\lambda_k\|_{l^p} \lesssim \|F\|_{L^p S} \tag{1.12}$$

where  $F_k$  are  $p$ -atoms.

*Proof.* It is straightforward to check that

$$\sum_{k \in \mathbb{Z}} 2^k \mu(\|F\|_S > 2^{k/p}) \approx \|F\|_{L^p S}^p < \infty$$

Choose subsets  $E_k \subset X$  such that

$$\|\mathbb{1}_{\mathbb{X} \setminus E_k} F\|_S \leq 2^{k/p} \quad \mu(E_k) \lesssim \mu(\|F\|_S > 2^{k/p})$$

and thus  $\sum_{k \in \mathbb{Z}} 2^k \mu(E_k) \approx \|F\|_{L^p S}^p$ . We can actually assume that the sets  $E_k$  are decreasing. As a matter of fact

Set  $\tilde{E}_k = \bigcup_{l \geq k} E_l$  so it holds that

$$\|\mathbb{1}_{\mathbb{X} \setminus \tilde{E}_k} F\|_S \leq \|\mathbb{1}_{\mathbb{X} \setminus E_k} F\|_S \leq 2^{k/p}.$$

while

$$\sum_{k \in \mathbb{Z}} 2^k \mu(\tilde{E}_k) \lesssim \sum_{k \in \mathbb{Z}} 2^k \sum_{l \geq k} \mu(E_l) \lesssim \sum_{l \in \mathbb{Z}} 2^l \mu(E_l) \sum_{k \leq l} 2^{(k-l)} \approx \|F\|_{L^p S}^p.$$

Now consider the sets  $\Delta E_k = \tilde{E}_k \setminus \tilde{E}_{k+1}$  so that

$$\mathbb{X} = (\mathbb{X} \setminus \tilde{E}_{-\infty}) \cup \tilde{E}_{+\infty} \cup \left( \bigcup_{k \in \mathbb{Z}} \Delta E_k \right).$$

where  $\tilde{E}_{+\infty} = \bigcap_{k \in \mathbb{Z}} \tilde{E}_k$  and  $\tilde{E}_{-\infty} = \bigcup_{k \in \mathbb{Z}} E_k$ . Clearly  $\mu(\tilde{E}_{+\infty}) = 0$  while  $\|F \mathbb{1}_{\mathbb{X} \setminus \tilde{E}_{-\infty}}\|_S = 0$  so  $F \mathbb{1}_{\mathbb{X} \setminus \tilde{E}_{-\infty}} = F \mathbb{1}_{\tilde{E}_{+\infty}} = 0$ , at least as equivalence classes in  $\mathcal{B}(\mathbb{X})$ .

We may thus write

$$F = \sum_k \|F \mathbb{1}_{\Delta E_k}\|_S \mu(\Delta E_k)^{1/p} \frac{F \mathbb{1}_{\Delta E_k}}{\|F \mathbb{1}_{\Delta E_k}\|_S \mu(\Delta E_k)^{1/p}} = \sum_k \lambda_k F_k$$

and since  $\|F \mathbb{1}_{\Delta E_k}\|_S \lesssim 2^{k/p}$  it holds that

$$\|\lambda_k\|_{l^p}^p \lesssim \sum_k 2^k \mu(\Delta E_k)^{1/p} \lesssim \|F\|_{L^p S}^p.$$

It is also clear that  $F_k$  are  $p$ -atoms. □

Let  $\Lambda$  be a sub-linear form on  $\mathcal{B}(\mathbb{X})$  that is uniformly bounded on 1-atoms. It follows from the above decomposition that it is bounded on  $F_{L^1 S}$  i.e. it satisfies  $|\Lambda(F)| \lesssim \|F\|_{L^1 S}$ .

For outer measures and sizes generated as in (1.1) and (1.4) we have the following differentiation property.

**Proposition 1.9** (Measure differentiation). *Suppose that  $\Lambda$  is a sub-linear functional on Borel functions on an outer measure space  $(\mathbb{X}, \mu)$  i.e.*

$$|\Lambda(F + G)| \leq |\Lambda(F)| + |\Lambda(G)| \quad \text{and} \quad \Lambda(\lambda F) = |\lambda| \Lambda(F),$$

and suppose that  $\Lambda$  is quasi lower semi-continuous with respect to pointwise convergence i.e.

$$|\Lambda(F)| \lesssim \liminf |\Lambda(F_n)| \quad \text{if } F_n \rightarrow F \text{ pointwise.}$$

Let the size  $\|\cdot\|_S$  be generated by the family of sizes  $\|\cdot\|_{S(T)}$  with  $T \in \mathbb{T}$  while  $\mu$  is generated by  $\bar{\mu} : \mathbb{T} \rightarrow [0, \infty]$ . If it holds that

1. for every  $T \in \mathbb{T}$  it holds that

$$|\Lambda(F \mathbb{1}_T)| \lesssim \bar{\mu}(T) \|F\|_{S(T)}, \tag{1.13}$$

2. for any  $E \subset \mathbb{X}$  it holds that

$$\mu(E) = 0 \implies |\Lambda(F \mathbb{1}_E)| = 0,$$

3.  $\mathcal{X}$  is  $\sigma$ -finite as in (1.2).

then for all  $F \in L^1 S$

$$|\Lambda(F)| \lesssim \|F\|_{L^1 S} \quad (1.14)$$

holds.

Condition 2 can be dropped if  $\|F\|_S < \infty$ .

*Proof.* We suppose that  $\|F\|_{L^1 S}$  is finite, otherwise there is nothing to prove. Consider the 1-atomic decomposition of

$$F = \sum_{k \in \mathbb{Z}} \lambda_k F_k \quad \sum_{k \in \mathbb{Z}} |\lambda_k| \lesssim \|F\|_{L^1 S},$$

where this equality holds pointwise as equivalence classes in  $\mathcal{B}(\mathcal{X})$ . We postpone checking that  $\Lambda$  is well defined for equivalence classes in  $\mathcal{B}(\mathcal{X})$ . It is sufficient to check that (1.14) holds for 1-atoms since by subadditivity and lower semi-continuity one has

$$|\Lambda(F)| \leq \sum_{k \in \mathbb{Z}} |\lambda_k| |\Lambda(F_k)| \lesssim \|F\|_{L^1 S}$$

as long as  $|\Lambda(F_k)| \lesssim 1$ . Given an atom  $A$  let us choose a covering  $(T_n \in \mathbb{T})_{n \in \mathbb{N}}$  such that

$$\sum_{n \in \mathbb{N}} \bar{\mu}(T_n) \lesssim \mu(\text{spt } A) \quad \text{spt } A \subset \bigcup_{n \in \mathbb{N}} T_n.$$

Using (1.13) we have that

$$|\Lambda(F)| \leq \sum_{n \in \mathbb{N}} |\Lambda(F \mathbf{1}_{T_n \setminus \bigcup_{k > n} T_k})| \lesssim \sum_{n \in \mathbb{N}} \bar{\mu}(T_n) \|F \mathbf{1}_{T_n \setminus \bigcup_{k > n} T_k}\|_S \lesssim \mu(\text{spt } A) \|F\|_S = 1$$

as required.

We now check that  $\Lambda$  is well defined for functions in  $F \in \mathcal{B}(\mathcal{X})$ . By sub-linearity it is sufficient to check that  $\Lambda(F) = 0$  if  $\mu(\text{spt } F) = 0$ . Notice that if  $\mu(\text{spt } F) = 0$  then there exists a set  $K \subset \mathcal{X}$  with  $\mu(K) = 0$  and  $\|F \mathbf{1}_{\mathcal{X} \setminus K}\|_S = 0$ . As a matter of fact for  $n \in \mathbb{N}$  let  $K_n \subset \mathcal{X}$  such that  $\mu(K_n) \leq 2^{-n}$  and such that  $\|F \mathbf{1}_{\mathcal{X} \setminus K_n}\|_S = 0$ . Setting  $K = \bigcap_{k \in \mathbb{N}} \bigcup_{n > k} K_n$  implies that  $\mu(K) = 0$  and by property (1.3) it holds that  $\|F \mathbf{1}_{\mathcal{X} \setminus K}\|_S = 0$ . Since  $|\Lambda(F)| \leq |\Lambda(F \mathbf{1}_K)| + |\Lambda(F \mathbf{1}_{\mathcal{X} \setminus K})|$  we show that both terms on the right vanish.

Since  $\mathcal{X}$  is  $\sigma$ -finite let  $(T_n \in \mathbb{T})$  be a covering with  $\bar{\mu}(T_n) < \infty$  and  $\bigcup_{n \in \mathbb{N}} T_n = \mathcal{X}$ . Then

$$|\Lambda(F \mathbf{1}_{\mathcal{X} \setminus K})| \lesssim \sum_{n \in \mathbb{N}} |\Lambda(F \mathbf{1}_{T_n \setminus K})| \lesssim \sum_{n \in \mathbb{N}} \bar{\mu}(T_n) \|F \mathbf{1}_{T_n \setminus K}\|_S = 0$$

as required.

By 2 we have that

$$|\Lambda(F \mathbf{1}_K)| = 0.$$

The condition 2 can be dropped if  $\|F\|_S < \infty$  since, as before

$$|\Lambda(F \mathbf{1}_K)| \lesssim \liminf_{n \in \mathbb{N}} |\Lambda(F \mathbf{1}_{K_n})| \lesssim \liminf_{n \in \mathbb{N}} \mu(K_n) \|F\|_S = 0.$$

□

If  $\Lambda$  is an integral with respect to a Borel measure we have the following corollary.



**Corollary 1.10.** *Let  $\mathcal{L}$  is a non-negative Borel measure on  $\mathbb{X}$  and for every  $T \in \mathbb{T}$  let*

$$\|F\|_{S(T)} = \frac{1}{\bar{\mu}(T)} \int_T |F(x)| d\mathcal{L}(x)$$

*then if either  $\mu(E) = 0 \implies \mathcal{L}(E) = 0$  for all  $E \subset \mathbb{X}$  or  $\|F\|_S < \infty$  then*

$$\int_{\mathbb{X}} |F(x)| d\mathcal{L}(x) \lesssim \|F\|_{L^1 S}.$$

## 1.2 Classical Carleson embeddings

The framework of outer measure  $L^p$  spaces allows one to obtain some classical results from Calderón-Zygmund theory. These results rely on studying the classical (time-scale) Carleson embedding for  $L^p$  functions given by

$$F(y, t) = f * \psi_t(y) \quad \psi_t(x) := t^{-1} \psi\left(\frac{x}{t}\right) \quad (1.15)$$

with  $\psi$  some base wavelet. For example if

$$\psi(x) = \frac{1}{\pi} \frac{1}{x^2 + 1}$$

$F$  becomes the Poisson extention into the upper half-plane. Similarly if

$$\psi(x) = \frac{1}{\pi} \frac{x}{x^2 + 1}$$

then  $F$  becomes the harmonic conjugate to the Poisson extention.

Embedding maps actually give a faithful representation of a function as testified by Calderón reproducing formula. Let  $\psi, \psi^* \in S(\mathbb{R})$  be any two functions such that  $\int \psi = \int \psi^* = 0$  and  $\widehat{\psi} \widehat{\psi^*}$  is an even, real-valued, non-negative function. Then for any function  $f \in \dot{S}(\mathbb{R})$  the following identity holds pointwise and in  $L^2$ :

$$f(x) = C_\psi \int_0^{+\infty} f * \psi_t * \psi_t^*(x) \frac{dt}{t} \quad (1.16)$$

where  $C_\psi = \int_0^{+\infty} \widehat{\psi}(t) \widehat{\psi^*}(t) \frac{dt}{t}$  is some constant that depends only on  $\psi$ . Using the Fourier transform we have that

$$\begin{aligned} \mathcal{F}\left(\int_\varepsilon^{\varepsilon^{-1}} f * \psi_t * \psi_t^*(x) \frac{dt}{t}\right) &= \int_\varepsilon^{\varepsilon^{-1}} \widehat{f}(\xi) \widehat{\psi}(t\xi) \widehat{\psi^*}(t\xi) \frac{dt}{t} \\ &= \int_{|\xi| \leq \varepsilon^{-1}} \widehat{f}(\xi) \widehat{\psi}(\text{sign}(\xi)\tau) \widehat{\psi^*}(\text{sign}(\xi)\tau) \frac{d\tau}{\tau}. \end{aligned} \quad (1.17)$$

In the last equality we changed variables  $\tau = |\xi|t$ . Since  $\widehat{\psi}(0) = \widehat{\psi^*}(0) = 0$  we have that

$$C_\psi = \int_{\mathbb{R}} \widehat{\psi}(\tau) \widehat{\psi^*}(\tau) \frac{d\tau}{\tau} = \int_{\mathbb{R}} \widehat{\psi}(-\tau) \widehat{\psi^*}(-\tau) \frac{d\tau}{\tau} \in (0, \infty).$$

By dominated convergence the integral in (1.17) converges to  $C_\psi \widehat{f}(\xi)$  as  $\varepsilon \rightarrow 0$ . This concludes the proof of (1.16).

Outer-measure  $L^p$  are the appropriate norms for studying the embedding maps (1.15). The embedded function  $F$  is a Borel function on  $\mathbb{R}_+^2 = \mathbb{R} \times \mathbb{R}_+$ . We recall that the outer measure  $\mu$  on  $\mathbb{R}_+^2$  is generated by setting

$$\bar{\mu}(T(x, s)) = s \quad \text{with } T(x, s) = \{(y, t) \in \mathbb{R}_+^2 : |y - x| < s - t, t < s\}.$$

On  $\mathbb{R}_+^2$  we introduce the following family of sizes:

$$\|F\|_{S^r(T(x,s))} := \left( \frac{1}{s} \int_{T(x,s)} |F(y, t)|^r dy \frac{dt}{t} \right).$$

Then the following embedding Theorem holds.

**Theorem 1.11** (Carleson embedding theorem). *The embedding map (1.15) satisfies the bounds*

$$\|F\|_{L^p S^\infty} \lesssim \|f\|_{L^p} \quad p \in (1, \infty]$$

Furthermore if  $\int \psi = 0$  then it also satisfies the bound

$$\|F\|_{L^p S^2} \lesssim \|f\|_{L^p} \quad p \in (1, \infty]$$

*Proof.* We may restrict ourselves to proving the bound for  $p = \infty$  and the weak bound for  $p = 1$ . The full result then follows by interpolation as in Proposition 1.7 applied between classical and outer-measure  $L^p$  spaces.

**Case  $\int \psi \neq 0$ :**

To show the bound  $\|F\|_{L^\infty S^\infty} \lesssim \|f\|_{L^\infty}$  it is sufficient to show that for any tent  $T(x, s)$  we have

$$\|F\|_{S^\infty(T)} = \sup_{(y,t) \in T(x,s)} |F(y, t)| \lesssim \|f\|_{L^\infty}.$$

This is trivial since

$$|F(y, t)| = |f * \psi_t(y)| \leq \|f\|_{L^\infty} \|\psi_t\|_{L^1} \lesssim \|f\|_{L^\infty}.$$

To show the bound  $\|F\|_{L^{1,\infty} S^\infty} \lesssim \|f\|_{L^1}$  we must show that for every  $\lambda > 0$  there exists a set  $E_\lambda \subset \mathbb{R}_+^2$  such that

$$\mu(E_\lambda) \lesssim \frac{\|f\|_{L^1}}{\lambda} \quad \|F \mathbb{1}_{\mathbb{R}_+^2 \setminus E_\lambda}\|_{S^\infty} \lesssim \lambda.$$

To do so let us consider the open set

$$\{x : Mf(x) > \lambda\} = \bigcup_{n \in \mathbb{N}} B_{s_n}(x_n)$$

where  $B_{s_n}(x_n)$  is a (maximal) covering using disjoint open balls. Let us set

$$E_\lambda = \bigcup_{n \in \mathbb{N}} T(x_n, 3s_n) \quad \implies \mu(E_\lambda) \leq \sum_{n \in \mathbb{N}} \bar{\mu}(T(x_n, 3s_n)) \leq \sum_{n \in \mathbb{N}} 3s_n \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

The estimate on the measure comes from the boundedness of the Hardy-Littlewood Maximal function. It remains to check that

$$\|F \mathbb{1}_{E_\lambda^c}\|_{S^\infty} \lesssim \lambda.$$

This follows by contradiction. Suppose that for any  $C > 1$  there exists some  $(y, t) \notin E_\lambda$  such that

$$|f| * \psi_t(y) > C\lambda.$$

Then, if  $C$  is large enough,  $Mf(y') > \lambda$  for all  $|y' - y| < t$  and thus  $B_t(y) \subset B_{s_n}(x_n)$  for some  $n \in \mathbb{N}$  but this contradicts that  $(y, t) \notin E_\lambda$ .

**Case  $\int \psi = 0$ :**

We will restrict to the case where  $\text{spt } \psi \subset [-1, 1]$ . This is a non-essential restriction: if  $\psi \in S(\mathbb{R})$  for an arbitrarily large  $N > 0$  one can decompose

$$\psi(x) = \sum_{n=0}^{\infty} 2^{-Nn} \psi_n\left(\frac{x}{2^n}\right)$$

where  $\text{spt } \psi_n \in B_1$  are uniformly bounded Schwartz functions with  $\int \psi_n = 0$ . It is sufficient to use quasi subadditivity of outer  $L^p S^2$  norms.

First let us start by showing the bounds  $\|F\|_{L^\infty S^2} \lesssim \|f\|_{L^\infty}$  i.e. we must show that for any tent  $T(x, s)$

$$\frac{1}{s} \iint_{T(x,s)} |f * \psi_t(y)|^2 dy \frac{dt}{t} \lesssim \|f\|_{L^\infty}^2$$

By using the Plancherel identity we have that

$$\begin{aligned} \frac{1}{s} \iint_{T(x,s)} |f * \psi_t(y)|^2 dy \frac{dt}{t} &\leq \frac{1}{s} \iint_{\mathbb{R}_+^2} |f * \psi_t(y)|^2 dy \frac{dt}{t} \leq \frac{1}{s} \iint_{\mathbb{R}_+^2} |\widehat{f}(\xi)|^2 |\widehat{\psi}(t\xi)|^2 d\xi \frac{dt}{t} \\ &= \frac{1}{s} \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \int_{\mathbb{R}^+} |\widehat{\psi}(\text{sign}(\xi)\tau)|^2 \frac{d\tau}{\tau} d\xi \lesssim \frac{1}{s} \|f\|_{L^2}^2 \end{aligned}$$

where we used the change of variables  $\tau = |\xi|t$ . Notice that if  $(y, t) \in T(x, s)$  then

$$f * \psi_t(y) = \widetilde{f} * \psi_t(y)$$

with  $\widetilde{f} = f \mathbb{1}_{B_{3s}(x)}$  so

$$\frac{1}{s} \iint_{T(x,s)} |f * \psi_t(y)|^2 dy \frac{dt}{t} \lesssim \frac{1}{s} \|f \mathbb{1}_{B_{3s}(x)}\|_{L^2}^2 \lesssim \|f\|_{L^\infty}$$

as required.

Finally we proceed to show the weak  $L^1$  bounds  $\|F\|_{L^{1,\infty} S^2} \lesssim \|f\|_{L^1}$ . Recall that definition (1.6) for each  $\lambda > 0$  we need to find a set  $E_\lambda \subset \mathbb{R}_+^2$

$$\mu(E_\lambda) \lesssim \frac{\|f\|_{L^1}}{\lambda} \quad \|F \mathbb{1}_{\mathbb{R}_+^2 \setminus E_\lambda}\|_{S^2} \lesssim \lambda.$$

A Calderón-Zygmund decomposition of  $f$  at level  $\lambda$  allows us to write

$$f = g + b = g + \sum_n b_n$$

with  $\|g\|_{L^\infty} \lesssim \lambda$  and

$$\begin{aligned} \text{spt } b_n &= B_{s_n}(x_n) & \int_{\mathbb{R}} b_n &= 0 \\ \int_{B_{s_n}(x_n)} |b_n| &\lesssim \lambda & \sum_{n \in \mathbb{N}} |B_{s_n}(x_n)| &\lesssim \frac{\|f\|_{L^1}}{\lambda}. \end{aligned}$$

Set  $E_\lambda = \bigcup_{n \in \mathbb{N}} T(x_n, 3s_n)$  so that the condition on the outer measure of  $E_\lambda$  is satisfied. Notice that

$$F(y, t) = G(y, t) + B(y, t) = G(y, t) + \sum_n B_n(y, t) = g * \psi_t(y) + \sum_n b_n * \psi_t(y).$$

By the  $L^\infty$  bound above we have that

$$\|G\|_{S^2(T)} \lesssim \lambda$$

By quasi subadditivity it remains to show that for any tent  $T(x, s)$  it holds that

$$\begin{aligned} \|B\mathbb{1}_{\mathbb{R}_+^2 \setminus E_\lambda}\|_{S^2(T(x,s))}^2 &= \frac{1}{s} \iint_{T(x,s) \setminus E_\lambda} |b * \psi_t(y)|^2 dy \frac{dt}{t} \\ &\leq \iint_{T(x,s) \setminus E_\lambda} \left| \sum_{n: s_n \leq t} b_n * \psi_t(y) \right|^2 dy \frac{dt}{t} \lesssim \lambda^2 \end{aligned}$$

where the second-to-last inequality follows by the fact that  $\text{spt } \psi_t \subset [-t, t]$ . We will actually show that

$$\|B\mathbb{1}_{\mathbb{R}_+^2 \setminus E_\lambda}\|_{S^\infty(T(x,s))} \lesssim \lambda \quad \|B\mathbb{1}_{\mathbb{R}_+^2 \setminus E_\lambda}\|_{S^1(T(x,s))} \lesssim \lambda$$

from which the bound for  $\|\cdot\|_{S^2}$  follows.

As a matter of fact

$$t < s_j, (y, t) \notin T(x_j, 3s_j) \implies b_j * \psi_t(y) = 0.$$

Let  $\beta_n$  be the primitive of  $b_n$ , supported on  $B_{s_n}(x_n)$ . Integrating by parts one has

$$\left| \sum_{n: s_n < t} b_n * \psi_t(y) \right| \lesssim |t^{-1} \sum_{n: s_n < t} \beta_n * \psi'_t(y)| \lesssim |t^{-1} \sum_{n: s_n < t} \beta_n|_{L^\infty} \|\psi'_t\|_{L^1} \lesssim \lambda \quad (1.18)$$

since  $t^{-1} \|\beta_n\|_{L^\infty} \leq s_n^{-1} \|\beta_n\|_{L^\infty} \lesssim \lambda$  and the supports of  $\beta_n$  are disjoint.

Similarly for  $\|\cdot\|_{S^1}$  we write

$$\begin{aligned} \frac{1}{s} \iint_{T(x,s) \setminus E_\lambda} |b * \psi_t(y)| dy \frac{dt}{t} &\leq \sum_{n: s_n < t} \frac{1}{s} \iint_{T(x,s) \setminus E_\lambda} |b_n * \psi_t(y)| dy \frac{dt}{t} \\ &\leq \sum_{n: B_{s_n}(x_n) \cap B_{3s}(x) \neq \emptyset} \frac{1}{s} \int_{t > s_n} \int_{|y-x_n| < 2t} t^{-2} |\beta_n * \psi'_t(y)| dy \frac{dt}{t} \\ &\lesssim \sum_{n: B_{s_n}(x_n) \cap B_{3s}(x) \neq \emptyset} \frac{1}{s} \int_{t > s_n} \int_{|y-x_n| < 2t} t^{-2} \|\beta_n\|_{L^1} \|\varphi'\|_{L^\infty} dy \frac{dt}{t} \\ &\lesssim \frac{\lambda}{s} \sum_{n: B_{s_n}(x_n) \cap B_{3s}(x) \neq \emptyset} s_n^{-1} s_n^2 \lesssim \lambda \end{aligned} \quad (1.19)$$

This concludes the proof. □

We will now use the above embedding map to obtain several well known results from classical harmonic analysis.

### 1.2.1 The Hilbert transform

Using the Calderón reproducing formula we may prove the boundedness of the Hilbert transform on  $L^p(\mathbb{R})$ . Recall that

$$\mathcal{H}f(x) := \frac{1}{\pi} \text{PV} \int_{\mathbb{R}} f(x-y) \frac{dy}{y}.$$

Using Calderón's reproducing formula (1.16) with  $\psi^*$  chosen so that  $\widehat{\psi^*} = 0$  on  $B_\varepsilon(0)$  for some small  $\varepsilon > 0$  we have

$$\begin{aligned} \mathcal{H}f(\cdot) &= \mathcal{H} \left( \int_{\mathbb{R}_+^2} f * \psi_t(z) \psi_t^*(\cdot - z) dz \frac{dt}{t} \right) \\ &= \int_{\mathbb{R}_+^2} f * \psi_t(z) (\mathcal{H}\psi_t^*)(\cdot - z) dz \frac{dt}{t} = \int_{\mathbb{R}_+^2} f * \psi_t(z) \varphi_t(\cdot - z) dz \frac{dt}{t}. \end{aligned}$$

where  $\varphi(x) = \mathcal{H}\psi^*(x)$ . We used the scaling and translation invariance of the operator  $\mathcal{H}$ . Since  $0 \notin \text{spt} \widehat{\psi^*}$  i.e.  $\widehat{\psi^*}$  is supported away from the singularity of the multiplier  $-i \text{sign}(\xi)$  of  $\mathcal{H}$ , it holds that  $\varphi \in S(\mathbb{R})$  and  $\int \varphi = 0$ .

By duality the boundedness of  $\mathcal{H}$  on  $L^p(\mathbb{R})$  follows by showing

$$\begin{aligned} \left| \int_{\mathbb{R}} \mathcal{H}f(x)g(x)dx \right| &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}_+^2} f * \psi_t(z) \varphi_t(x-z)g(x) dz \frac{dt}{t} dx \right| \\ &= \left| \int_{\mathbb{R}_+^2} F(z,t)G(z,t) dz \frac{dt}{t} \right| \lesssim \|f\|_{L^p} \|g\|_{L^{p'}} \end{aligned} \tag{1.20}$$

for all functions  $f, g \in S(\mathbb{R})$ , where

$$F(z,t) = f * \psi_t(z) \quad G(z,t) = g * \varphi_t^\vee(z) \text{ with } \varphi^\vee(x) := \varphi(-x).$$

The embedding Theorem 1.11 allows us to conclude. As a matter of fact we have that

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} F(z,t)G(z,t) dz \frac{dt}{t} \right| &\lesssim \|F(z,t)G(z,t)\|_{L^1(S^1)} \quad \text{corollary 1.10} \\ &\lesssim \|F(z,t)\|_{L^p(S^2)} \|G(z,t)\|_{L^{p'}(S^2)} \quad \text{outer-measure Hölder 1.6} \\ &\lesssim \|f\|_{L^p} \|g\|_{L^{p'}} \quad \text{embedding Theorem 1.11.} \end{aligned}$$

The above procedure can be generalized to operators given by smooth Mihlin multipliers as long as one shows that the embedding Theorem 1.11 holds for

$$F(z,t) = \int_{\mathbb{R}} f(x) \varphi_{z,t}(x) dx$$

where  $\varphi_{z,t}$  is a family of wavelets indexed by  $(z,t) \in \mathbb{R}_+^2$  such that  $t\varphi_{z,t}(t \cdot + z)$  is uniformly bounded in  $S(\mathbb{R})$  and with that  $0 \notin \text{spt} \widehat{\varphi_{z,t}}$ . Non-smooth or non translation invariant operators are beyond the scope of this thesis.

### 1.2.2 Paraproducts

The simplest example of a paraproduct is a bilinear form

$$\mathcal{P}(f,g)(x) = \int_0^{+\infty} f * \varphi_t(x) g * \psi_t(x) \frac{dt}{t}$$

where  $\varphi, \psi \in S(\mathbb{R})$  with  $\int \psi = 0$ . Paraproducts are closely related to products. As a matter of fact given two functions  $f, g \in S(\mathbb{R})$  and  $\varphi \in S(\mathbb{R})$  with  $\int \varphi = 1$  we have that

$$\begin{aligned} f(x)g(x) &= \lim_{t \rightarrow 0} f * \varphi_t(x) g * \varphi_t(x) = - \int_0^{+\infty} \partial_t (f * \varphi_t(x) g * \varphi_t(x)) dt \\ &= - \int_0^{+\infty} f * \partial_t \varphi_t(x) g * \varphi_t(x) dt - \int_0^{+\infty} f * \varphi_t(x) g * \partial_t \varphi_t(x) dt \\ &= \int_0^{+\infty} f * \psi_t(x) g * \varphi_t(x) \frac{dt}{t} + \int_0^{+\infty} f * \varphi_t(x) g * \psi_t(x) \frac{dt}{t} \end{aligned}$$

where  $\psi(x) = -\partial_x(x\varphi(x))$  and so  $\int \psi = 0$ . The second equality above holds since  $f * \varphi_t(x) \rightarrow 0$  as  $t \rightarrow +\infty$ . We used that

$$\partial_t \varphi_t(x) = -t^{-2} \varphi(t^{-1}x) - t^{-3} x \varphi'(t^{-1}x) = -\frac{1}{t} \psi_t(x).$$

Thus we have obtained that

$$f(x)g(x) = \mathcal{P}(f, g) + \mathcal{P}(g, f).$$

By the classical Hölder inequality we have that

$$\|fg\|_{L^{p'_3}} \lesssim \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} \quad 1 - \frac{1}{p'_3} = \frac{1}{p_1} + \frac{1}{p_2} \quad p_{1,2} \in [1, \infty], p'_3 \in [1, \infty].$$

Let us show that the paraproduct  $\mathcal{P}$  also satisfies these Hölder-type bounds (except for the endpoints).

We make the inessential assumption that  $\text{spt } \widehat{\psi} \subset \{\xi : 1 < |\xi| < 2\}$  instead of simply  $\widehat{\psi}(0) = 0$  and that  $\text{spt } \widehat{\varphi} \subset \{\xi : |\xi| < 2\}$ . This simplifies some of the technicalities of the argument.

The trilinear for dual to  $\mathcal{P}$  is given by

$$(f_1, f_2, f_3) \mapsto \int_{\mathbb{R}} \mathcal{P}(f_1, f_2)(x) f_3(x) dx = \int_{\mathbb{R}} \int_{\mathbb{R}^+} f_1 * \varphi_t(x) f_2 * \psi_t(x) \frac{dt}{t} f_3(x) dx$$

Applying the Fourier transform and commuting the integrals we have that

$$\begin{aligned} & \int_{\mathbb{R}} \mathcal{P}(f_1, f_2)(x) f_3(x) dx \\ &= \int_{\mathbb{R}} \int_0^\infty \left( \int_{\mathbb{R}} \widehat{f}_1(\xi_1) e^{2\pi i \xi_1 x} \widehat{\varphi}(t\xi_1) d\xi_1 \int_{\mathbb{R}} \widehat{f}_2(\xi_2) e^{2\pi i \xi_2 x} \widehat{\psi}(t\xi_2) d\xi_2 \right) \frac{dt}{t} f_3(x) dx \\ &= \int_0^\infty \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(-\xi_1 - \xi_2) \widehat{\varphi}(t\xi_1) \widehat{\psi}(t\xi_2) d\xi_1 d\xi_2 \frac{dt}{t} \\ &= \int_0^\infty \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(-\xi_1 - \xi_2) \widehat{\varphi}^{(1)}(t\xi_1) \widehat{\psi}(t\xi_2) d\xi_1 d\xi_2 \frac{dt}{t} \\ &+ \int_0^\infty \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(-\xi_1 - \xi_2) \widehat{\psi}^{(1)}(t\xi_1) \widehat{\varphi}(t\xi_2) d\xi_1 d\xi_2 \frac{dt}{t}. \end{aligned}$$

where we decomposed  $\varphi = \varphi^{(1)} + \psi^{(1)}$  with  $\varphi^{(1)}, \psi^{(1)} \in S(\mathbb{R})$  such that

$$\text{spt } \widehat{\varphi}^{(1)} \subset \{\xi : |\xi| < 1/2\} \quad \text{spt } \widehat{\psi}^{(1)} \subset \{\xi : 1/4 < |\xi| < 2\}.$$

Now let us introduce a  $\varphi^{(3)}, \psi^{(3)} \in S(\mathbb{R})$  such that

$$\widehat{\psi}^{(3)}(0) = 0, \quad \widehat{\psi}^{(3)}(\xi) = 1 \text{ on } \left( \text{spt } \widehat{\varphi}^{(1)} + \text{spt } \widehat{\psi} \right) \subset \{\xi: 1/2 < |\xi| < 4\},$$

and

$$\widehat{\varphi}^{(3)}(\xi) = 1 \text{ on } \left( \text{spt } \widehat{\psi}^{(1)} + \text{spt } \widehat{\psi} \right) \subset \{\xi: |\xi| < 4\}$$

Using the above conditions and inverting the Fourier transform we obtain that

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(-\xi_1 - \xi_2) \widehat{\varphi}^{(1)}(t\xi_1) \widehat{\psi}(t\xi_2) d\xi_1 d\xi_2 \frac{dt}{t} \\ = & \int_0^\infty \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(-\xi_1 - \xi_2) \widehat{\varphi}^{(1)}(t\xi_1) \widehat{\psi}(t\xi_2) \widehat{\psi}^{(3)}(-t(\xi_1 + \xi_2)) d\xi_1 d\xi_2 \frac{dt}{t} \\ & = \int_0^\infty \int_{\mathbb{R}} f_1 * \varphi_t^{(1)}(x) f_2 * \psi_t(x) f_3 * \psi_t^{(3)}(x) dx \frac{dt}{t} \end{aligned}$$

and

$$\begin{aligned} & \int_0^\infty \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(-\xi_1 - \xi_2) \widehat{\psi}^{(1)}(t\xi_1) \widehat{\psi}(t\xi_2) d\xi_1 d\xi_2 \frac{dt}{t} \\ = & \int_0^\infty \int_{\mathbb{R}^2} \widehat{f}_1(\xi_1) \widehat{f}_2(\xi_2) \widehat{f}_3(-\xi_1 - \xi_2) \widehat{\psi}^{(1)}(t\xi_1) \widehat{\psi}(t\xi_2) \widehat{\varphi}^{(3)}(-t(\xi_1 + \xi_2)) d\xi_1 d\xi_2 \frac{dt}{t} \\ & = \int_0^\infty \int_{\mathbb{R}} f_1 * \psi_t^{(1)}(x) f_2 * \psi_t(x) f_3 * \varphi_t^{(3)}(x) dx \frac{dt}{t} \end{aligned}$$

For the first term we use corollary 1.10 and the outer-measure Hölder inequality to obtain

$$\begin{aligned} & \left| \int_0^\infty \int_{\mathbb{R}} f_1 * \varphi_t^{(1)}(x) f_2 * \psi_t(x) f_3 * \psi_t^{(3)}(x) dx \frac{dt}{t} \right| \\ & \lesssim \|f_1 * \varphi_t^{(1)}(x) f_2 * \psi_t(x) f_3 * \psi_t^{(3)}(x)\|_{L^1 S^1} \\ & \lesssim \|f_1 * \varphi_t^{(1)}(x)\|_{L^{p_1} S^\infty} \|f_2 * \psi_t(x)\|_{L^{p_2} S^2} \|f_3 * \psi_t^{(3)}(x)\|_{L^{p_3} S^2} \\ & \lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}}. \end{aligned}$$

The last inequality follows from the embedding Theorem 1.11. The crucial observation is that of the three term in the integral, at least two appear convolved with a mean-zero functions  $\psi$ . The second term can be dealt with in the same way.

This approach can also be expanded to include more general paraproducts given by multipliers, but this is beyond the scope of this presentation

### 1.2.3 Sparse operators and Carleson measures

The classical notion of a Carleson measure  $\sigma$  on  $\mathbb{R}_+^2$  is a Borel measure that satisfies the following bound

$$\sigma(T(x, s)) \leq Cs \quad \forall (x, s) \in \mathbb{R}_+^2.$$

If  $\sigma = \rho(x, t) dx \frac{dt}{t}$  is given by a non-negative density then the above condition becomes

$$\frac{1}{s} \int_{T(x, s)} \rho(x, t) dx \frac{dt}{t} \leq C$$

i.e.  $\sigma$  is a Carleson measure if

$$\|\rho\|_{L^\infty S^1} < C \quad \text{for some } C > 0. \quad (1.21)$$

Given a Carleson measure let us consider the form

$$(f_1, f_2) \mapsto \int_{\mathbb{R}_+^2} \left( \mathcal{f}_{B_t(x)} |f_1| \right) \left( \mathcal{f}_{B_t(x)} |f_2| \right) d\sigma(x, t).$$

This is a positive form so we may assume that  $f_1$  and  $f_2$  are non-negative and it follows that  $\mathcal{f}_{B_t(x)} |f| \lesssim F(x, t) = f * \varphi_t(x)$  for some non-negative  $\varphi \in S(\mathbb{R})$ . Using the embedding 1.11 it follows that the above form satisfies the bounds

$$\int_{\mathbb{R}_+^2} \left( \mathcal{f}_{B_t(x)} |f_1| \right) \left( \mathcal{f}_{B_t(x)} |f_2| \right) d\sigma(x, t) \leq \|F_1\|_{L^p S^\infty} \|F_2\|_{L^{p'} S^\infty} \|\rho\|_{L^\infty S^1}$$

for all  $p, p' \in (1, \infty)$  with  $\frac{1}{p} + \frac{1}{p'} = 1$ .

Sparse forms are closely related to Carleson's measures. A discrete  $n$ -linear sparse form with exponents  $s_1, \dots, s_n \in (1, \infty]$  is given by

$$(f_1, \dots, f_n) \mapsto \sum_{I \in \mathfrak{S}} |I| \prod_{i=1}^n \left( \mathcal{f}_I |f_i|^{s_i} \right)^{1/s_i} \quad (1.22)$$

where  $\mathfrak{S}$  is a collection of intervals such that

$$\sum_{\substack{J \in \mathfrak{S} \\ J \subset I}} |J| \leq C|I|. \quad (1.23)$$

In a continuous setting we consider for which  $p_1, \dots, p_n \in (1, \infty]$  does the bound

$$\left| \int_{\mathbb{R}_+^2} \prod_{i=1}^n \left( \mathcal{f}_{B_t(x)} |f_i|^{s_i} \right)^{1/s_i} \rho(x, t) dx \frac{dt}{t} \right| \lesssim \prod_{i=1}^n \|f_i\|_{L^{p_i}} \quad (1.24)$$

with some  $\rho$  satisfying (1.21) hold. The relation between (1.22) and (1.24) is given by the fact that the former can be rewritten as

$$(f_1, \dots, f_n) \mapsto \int_{\mathbb{R}_+^2} \prod_{i=1}^n \left( \mathcal{f}_{B_t(x)} |f_i|^{s_i} \right)^{1/s_i} d\sigma(x, t)$$

where  $\sigma(x, t) = \sum_{I \in \mathfrak{S}} 2t \delta(x - x_I) \delta(t - |I|/2)$  and condition (1.23) implies that  $\sigma$  is a Carleson measure. Bounds (1.24) imply the bounds for (1.22) by a straightforward mollification procedure. The bound (1.24) follows by noticing that

$$\begin{aligned} \left| \int_{\mathbb{R}_+^2} \prod_{i=1}^n \left( \mathcal{f}_{B_t(x)} |f_i|^{s_i} \right)^{1/s_i} \rho(x, t) dx \frac{dt}{t} \right| &= \iint_{\mathbb{R}_+^2} \prod_{i=1}^n F_i(x, t) \rho(x, t) dx \frac{dt}{t} \\ &\lesssim \left\| \prod_{i=1}^n F_i(x, t) \rho(x, t) \right\|_{L^1 S^1} \lesssim \|\rho\|_{L^\infty S^1} \prod_{i=1}^n \|F_i\|_{L^{p_i} S^\infty} \lesssim \prod_{i=1}^n \|f_i\|_{L^{p_i}}. \end{aligned}$$

We have used the property 1.10 and the outer-measure Hölder inequality 1.6 that holds as long as  $\sum_{i=1}^n \frac{1}{p_i} = 1$ . The last inequality follows from the boundedness of the embedding map

$$F_i(x, t) = \left( \mathcal{f}_{B_t(x)} |f_i|^{s_i} \right)^{1/s_i}$$



that hold as long as  $p_i > s_i$ . This follows from Theorem 1.11 by noticing that

$$F_i(x, t)^{s_i} \lesssim |f_i|^{s_i} * \psi_t(x)$$

for some non-negative bump function  $\psi \in S(\mathbb{R})$  and that

$$\|F_i\|_{L^{p_i} S^\infty} \lesssim \|F_i^{s_i}\|_{L^{p_i/s_i} S^\infty}^{1/s_i} \lesssim \| |f_i|^{s_i} \|_{L^{p_i/s_i}}^{1/s_i} \lesssim \|f_i\|_{L^{p_i}}.$$

## 1.3 Time frequency analysis and the Walsh-Fourier model

### 1.3.1 The Walsh group

We now introduce the Walsh group that is a discrete analog of  $\mathbb{R}$ . The advantage of this analog is that many problems in Real Harmonic Analysis have a corresponding formulation for the Walsh group that maintains the major characteristics of the original problem while simplifying some of the more technical details. We describe the Walsh group in somewhat greater generality than strictly necessary in this application.

We begin by fixing a prime  $p$ , for our applications one may set  $p = 2$ . For every  $x \in \mathbb{R}_+$  consider its digit expansion  $(x_n)_{n \in \mathbb{Z}}$  in base  $p$  with  $x_n \in \{0, \dots, p-1\}$  and  $x = \sum_{n \in \mathbb{Z}} x_n p^n$  and instead of classical addition consider the operation  $\oplus$  that is digit-wise addition without carry. Formally the Walsh group with base  $p$  is obtained by setting

$$\begin{aligned} |x| &= \max\{p^n : x_n \neq 0\} & \text{for } x \in \prod_{n \in \mathbb{Z}} \mathbb{Z}_p \\ W_p &:= \left\{ x \in \prod_{n \in \mathbb{Z}} \mathbb{Z}_p : |x| < \infty \right\}. \end{aligned} \tag{1.25}$$

with  $\oplus$  induced by the sum on  $\mathbb{Z}_p$  i.e.

$$(x \oplus y)_n = (x_n + y_n) / \mathbb{Z}_p = \begin{cases} x_n + y_n & \text{if } x_n + y_n \leq p-1 \\ x_n + y_n - p & \text{if } x_n + y_n \geq p. \end{cases} \tag{1.26}$$

Both the 0 element and the inverse  $-x$  are well defined. The map  $(x, y) \mapsto |x - y|$  defines a translation invariant distance with respect to which  $W_p$  is complete. The topology is generated by  $p$ -adic intervals that are balls in the Walsh metric  $B_{p^n}(x) = \{y : |x - y| < p^n\}$  with  $x \in W_p$  and  $n \in \mathbb{Z}$ .

Intuitively, by defining  $\oplus$  this way, we are postulating that different orders of magnitude, represented by different digits, behave independently. Following the same intuition, multiplication on  $W_p$  is given by the Cauchy product

$$(x \odot y)_n = \left( \sum_{k+l=n} x_k y_l \right) / \mathbb{Z}_p$$

that is well defined since  $|x|, |y| < \infty$  and the norm  $x \mapsto |x|$  is multiplicative i.e.  $|xy| = |x||y|$ . By a slight abuse of notation for any  $n \in \mathbb{Z}$  set  $p^n \in W_p$  be given by

$$(p^n)_k = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

The multiplicative unit is then given by  $p^0 = 1$  and any non-zero number  $x \in W_p$  has an inverse. Given  $|y| < 1$  it holds that

$$(1 + y)^{-1} = \sum_{k=0}^{\infty} (-1)^k \underbrace{y \odot \cdots \odot y}_{k \text{ times}} = \sum_{k=0}^{\infty} (-1)^k \odot y^{\odot k}$$

while if  $a \in \{0, \dots, p-1\}$  the inverse  $a^{-1}$  in  $\mathbb{Z}_p$  exists since  $p$  is prime. Since for any  $x \in W_p$  it holds that  $x = p^n \odot (x_n) \odot (1 + p^{-n} \odot (x_n)^{-1}x)$  with  $p^n = |x|$  we can compute  $x^{-1}$  as

$$x^{-1} = p^{-n}(x_n)^{-1} \sum_{k=0}^{\infty} (-1)^k \odot (1 + p^{-n} \odot (x_n)^{-1}x)^{\odot k}.$$

From now on we will silently omit the  $\odot$  notation and write  $xy$  instead of  $x \odot y$  for  $x, y \in W_p$ . We will be using a crucial geometric remark about Walsh intervals: given two intervals  $I, I' \subset W_p$  it holds that

$$I \cap I' \neq \emptyset \implies I \subseteq I' \text{ or } I' \subseteq I.$$

Given a  $p$ -adic interval  $I = B_{|I|}(x) \subset W_p$  we say  $I'$  is a  $p$ -adic sibling of  $I$  if  $I' \subset B_{p|I|}(x) = B_{p|I'|}(x)$ . Given an interval  $I$  we denote by  $pI$  its  $p$ -adic parent i.e. if  $I = B_{|I|}(x)$  then  $pI = B_{p|I|}(x)$ .

Let us introduce a translation invariant (Haar) measure  $|\cdot|$  on  $W_p$  by setting  $|B_1(0)| = 1$  so that

$$|B_{p^n}(x)| = p^n$$

while the measure of other Borel sets can be determined via a covering argument. Notice that the map

$$x \mapsto \mathbf{i}(x) = \sum_{n \in \mathbb{Z}} x_n p^n. \quad (1.27)$$

from  $W_p$  to  $\mathbb{R}_+$  is continuous, surjective, and injective when restricted to the complement of the Walsh numbers of the form

$$\{x : x_n = p-1 \ \forall n < -N \text{ for some } N \in \mathbb{Z}\}.$$

This last set is countable so it has vanishing measure, and furthermore the above map  $\mathbf{i}$  is measure-preserving. In terms of function spaces  $f \mapsto f \circ \mathbf{i}$  is thus an isometry between  $L^p(\mathbb{R}_+)$  and  $L^p(W_p)$ .

Given two intervals  $I, I'$  we say that  $I$  is lower than  $I'$  ( $I'$  higher than  $I$  respectively) if all the points of the real interval  $\mathbf{i}(I)$  are smaller than all the points of  $\mathbf{i}(I')$ .

To introduce the Walsh-Fourier transform we introduce the shorthand

$$\exp(x) = e^{\frac{2\pi i}{p} x^{-1}}.$$

Given an integrable function  $f$  on  $W_p$  its Walsh-Fourier transform is defined by

$$\widehat{f}(\xi) := \int_{W_p} f(x) \exp(-\xi x) dx$$

where the integration is taken with respect to the measure we introduced on  $W_p$ . In terms of Fourier analysis on locally compact Abelian groups the dual group  $\widehat{W}_p$  of  $W_p$  can be identified with  $W_p$  itself via

$$x \mapsto \exp(\xi x) \quad \text{for some } \xi \in W_p.$$

Figure 1.2 illustrates the graphs of the characters of the group  $W_2$ . The numbers are represented in binary expansion

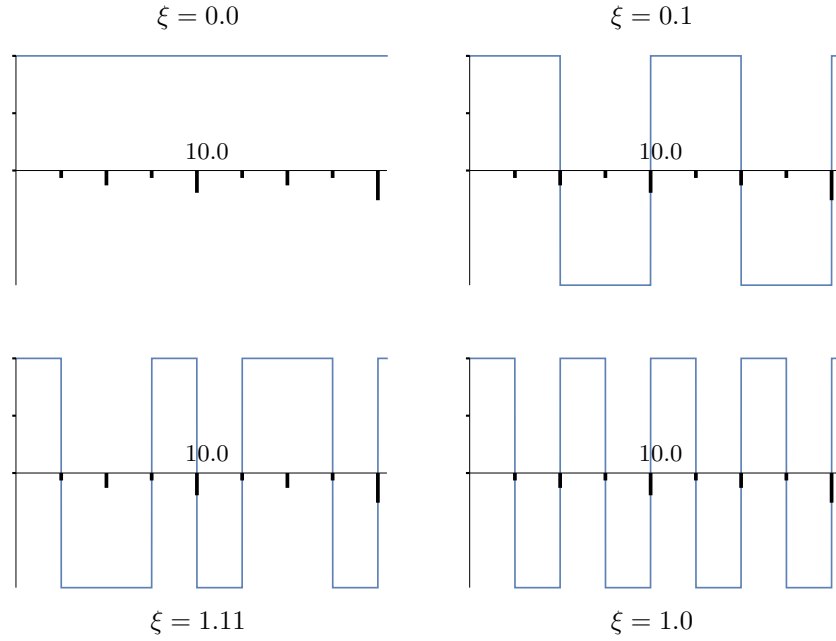


Figure 1.2: Some characters of the group  $W_2$ : graphs of the functions  $x \mapsto \exp(\xi x)$ .

**Proposition 1.12.** *The Hausdorff-Young inequality*

$$\|\widehat{f}\|_{L^{p'}(W_p)} \lesssim \|f\|_{L^p(W_p)} \quad p \in [1, 2]$$

and the Plancherel identity

$$\int_{W_p} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi = \int_{W_p} f(x) \overline{g(x)} dx \quad \forall f, g \in L^2(W_p) \quad (1.28)$$

hold.

Considering the translation, modulation, and dilation symmetries applied to an integrable function  $f$  we have the following identities that can be checked by direct computation

$$\begin{aligned} \widehat{f(\cdot - y)} &= \exp(-y \cdot) \widehat{f}(\cdot) \\ \widehat{\exp(\eta \cdot) f(\cdot)} &= \widehat{f}(\cdot - \eta) \\ \widehat{f(p^{-n} \cdot)} &= p^n \widehat{f}(p^n \cdot). \end{aligned} \quad (1.29)$$

Given  $y, \eta \in W_p$  and  $n \in \mathbb{Z}$  let us consider the function

$$\exp(\eta \cdot) p^{-n} \mathbb{1}_{B_{p^n}(y)}(\cdot).$$

Notice that given  $y' \in B_{p^n}(y)$  and  $\eta' \in B_{p^{-n}}(\eta)$  it holds that

$$\exp(\eta' \cdot) p^{-n} \mathbb{1}_{B_{p^n}(y')}(\cdot) = \exp((\eta' - \eta)y) \exp(\eta \cdot) p^{-n} \mathbb{1}_{B_{p^n}(y)}(\cdot) \quad (1.30)$$

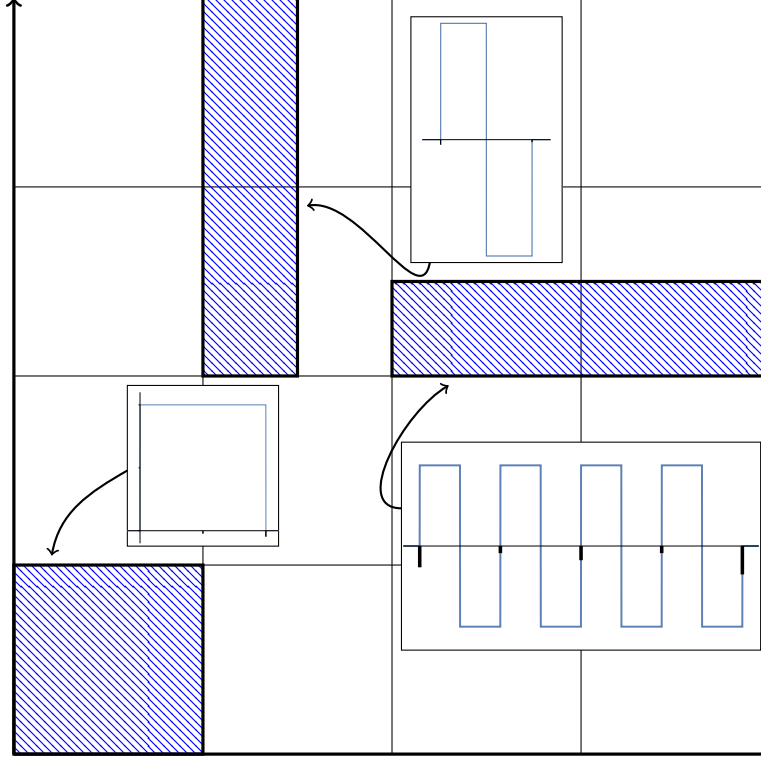


Figure 1.3: Walsh wave packets in the time-frequency plane.

i.e. the two functions differ by a factor  $\exp((\eta' - \eta)y)$  of norm 1 since for  $x \in B_{p^n}(y)$  it holds that  $\exp(\eta x) = \exp(\eta y)$ .

This motivates us to introduce the following notation. Let  $\mathbb{P}$  be the set of all  $p$ -adic rectangles of area 1 or a “tiles” i.e.

$$\mathbb{P} = \left\{ P = I_P \times \omega_P : I_P = B_{p^n}(y), \omega_P = B_{p^{-n}}(\eta) \text{ for some } y, \eta \in W_p \text{ and } n \in \mathbb{Z} \right\} \quad (1.31)$$

where  $I_P$  and  $\omega_P$  are the time and frequency projections of the tile. Notice that  $|I_P||\omega_P| = 1$ . Given a tile  $P$  we associate to it the wave packet

$$w_P(x) = |I_P|^{-1} \exp(\eta_P x) \mathbb{1}_{I_P}(x) \quad (1.32)$$

as in Figure 1.3.1. To avoid ambiguity we set  $\eta_P$  to be the “lower” point of  $\omega_P$  i.e.  $\eta_P \in \omega_P$  such that  $(\eta_P)_m = 0$  for all  $m \in \mathbb{Z}$  such that  $p^m < |\omega_P|$ .

The tiles encode the support of the wave packet and of its Walsh-Fourier transform.

**Proposition 1.13.** *Given a tile  $P \in \mathbb{P}$  and an associated wave packet  $w_P$  it holds that*

$$\text{spt } w_P = I_P \quad \text{spt } \widehat{w}_P = \omega_P.$$

Furthermore two wave packets  $w_P$  and  $w_{P'}$  are orthogonal if and only if  $P \cap P' = \emptyset$ , otherwise

$$\int_{W_p} w_P(x) \overline{w_{P'}(x)} dx = \max(|I_P|, |I_{P'}|)^{-1}.$$

Finally, any subset

$$\bigcup_n P_n = A \subset W_p \times W_p$$

where  $P_n$  are a finite collection of pairwise disjoint tiles identifies a finite-dimensional subspace of  $L^2(W_p)$  given by

$$\left\{ \sum_n a_n w_{P_n} : a_n \in \mathbb{C} \right\}.$$

This subspace depends only on  $A$  and not on the given representation of  $A$  as a union of tiles (tiling).

The proof of the above statements depends on the fact that

$$\widehat{\mathbb{1}_{B_1(0)}}(\xi) = \int_{B_1(0)} \exp(-\xi x) dx = \mathbb{1}_{B_1(0)}(\xi). \quad (1.33)$$

*Proof.* If  $|\xi| < 1$  and  $|x| < 1$  then  $\exp(\xi x) = 1$  so

$$\int_{B_1(0)} \exp(-\xi x) dx = \int_{B_1(0)} dx = 1.$$

On the other hand suppose  $|\xi| = p^m \geq 1$  so that we can represent  $B_1(0)$  as a union  $\bigcup_{i=1}^{p^m} B_{p^{-m}}(x_i)$  of pairwise disjoint  $p$ -adic intervals and  $\xi = \xi_m p^m + \xi_{<}$  with  $|\xi_{<}| < p^m$ . Thus using a change of variables and that  $\exp(-p^{-m} \xi_{<} x') = 1$  for  $x' \in B_1(0)$  we obtain that

$$\begin{aligned} \int_{B_1(0)} \exp(-\xi x) dx &= \sum_{i=1}^{p^m} \exp(-\xi x_i) \int_{B_{p^{-m}}(0)} \exp(-\xi_m p^m x) \exp(-\xi_{<} x) dx \\ &= \sum_{i=1}^{p^m} \exp(-\xi x_i) p^{-m} \int_{B_1(0)} \exp(-\xi_m x') dx' = 0 \end{aligned}$$

The last inequality follows from the fact that

$$\int_{B_1(0)} \exp(-\xi_m x') dx' = \sum_{q=0}^{p-1} \int_{B_{p^{-1}}(p^{-1}q)} \exp(-\xi_m x') dx = \sum_{q=0}^{p-1} p^{-1} e^{-\frac{2\pi i q \xi_m}{p}} = 0.$$

□

As a consequence, using (1.29) it follows that

$$\widehat{w_{I_P \times \omega_P}}(\xi) = w_{\omega_P \times I_P}(\xi). \quad (1.34)$$

If  $P$  and  $P'$  are disjoint either  $I_P \cap I_{P'} = \emptyset$  or  $\omega_P \cap \omega_{P'} = \emptyset$  so

$$\int_{W_p} w_P(x) \overline{w_{P'}(x)} dx = \int_{W_p} \widehat{w}_P(\xi) \overline{\widehat{w}_{P'}(\xi)} d\xi = 0.$$

If  $P$  and  $P'$  are not disjoint, we may, by applying a modulation and a translation to both functions assume that  $\omega_P, \omega_{P'}, I_P, I_{P'} \ni 0$  so

$$\int_{W_p} w_P(x) \overline{w_{P'}(x)} dx = |I_P|^{-1} |I_{P'}|^{-1} \int_{I_P \cap I_{P'}} dx = \max(|I_P|, |I_{P'}|)^{-1}.$$

The last part of Proposition 1.13 can first be proven in the special case

$$A = B_1(0) \times B_p(0) = \bigcup_{q=0}^{p-1} B_{p^{-1}}(p^{-1}q) \times B_p(0) = \bigcup_{q=0}^{p-1} B_1(0) \times B_1(q).$$

by counting dimensions. In fact the  $p$  wave packets in each representation are orthogonal and thus a basis, and the  $p \times p$  matrix

$$(q, q') \rightarrow \int w_{B_{p^{-1}}(p^{-1}q) \times B_p(0)}(x) \overline{w_{B_1(0) \times B_1(q')}(x)} dx = p^{-1} e^{\frac{2\pi i q q'}{p}}$$

is non-degenerate. One concludes the proof of 1.13 by an induction argument that we omit.

### 1.3.2 The Carleson operator

Let us now get back to the Carleson operator (0.1). Its linearized version is given by

$$\mathcal{C}_c f(z) = \int_{c(z)}^{\infty} \widehat{f}(\xi) e^{2\pi i \xi x} d\xi \quad (1.35)$$

for some fixed Borel measurable stopping function  $c : \mathbb{R} \rightarrow \mathbb{R}$ . To bound the above operator on  $L^p(\mathbb{R})$  we first need to obtain the wave-packet representation (0.3). We expect such a decomposition to be available since the family of operators (1.35) is essentially defined by its symmetries: apart from scaling and translation invariance typical of Calderón-Zygmund operators, the supremum in (0.1) accounts for modulation invariance i.e. if  $\tilde{f}(x) = e^{2\pi i \eta x} f(x)$  then  $\mathcal{C}f = \mathcal{C}\tilde{f}$ .

Identity (0.3) follows by duality from the equality

$$\mathcal{C}_c f(z) = \int_{\mathbb{R}_+^2} f * \psi_{\eta,t} * \psi_{\eta,t}(z) \chi(t(\eta - c(z))) d\eta dt \quad (1.36)$$

where

$$\psi_{\eta,t}(z) := t^{-1} e^{2\pi i \eta z} \psi\left(\frac{z}{t}\right) \quad (1.37)$$

with  $\psi \in S(\mathbb{R})$  a suitably normalized, non-negative, even, generating wavelet with Fourier transform  $\widehat{\psi}$  supported in a small ball  $B_b$ . The function  $\chi$  is a non-negative cutoff that satisfies

$$\chi \in C_c^\infty(B_\varepsilon(d)) \quad B_\varepsilon(d) \subset (b, +\infty) \quad \int \chi = 1 \quad (1.38)$$

and the truncated wave packet in (0.6) is given by

$$\psi_{\eta,t}^c(z) = \psi_{\eta,t}(z) \chi(t(\eta - c)). \quad (1.39)$$

*Proof of (1.36).* We now show that the above identity holds for  $f \in S(\mathbb{R})$  and for any bounded Borel function  $c$ . For every point  $z \in \mathbb{R}$  we introduce a modulated Littlewood-Paley frame centered at the frequency  $c(z) \in \mathbb{R}$  (compare to (1.20)). As a matter of fact freezing  $c = c(z)$  in

(1.36) and applying the Fourier transform gives

$$\begin{aligned}
& \mathcal{F} \left( \int_{\mathbb{R} \times \mathbb{R}^+} f * \psi_{\eta,t} * \psi_{\eta,t}(z) \chi(t(\eta - c)) d\eta dt \right) \\
&= \int_{\mathbb{R} \times \mathbb{R}^+} \widehat{f}(\xi) \widehat{\psi}(t(\xi - \eta))^2 t \chi(t(\eta - c)) d\eta \frac{dt}{t} \\
&= \int_{B_\varepsilon(d) \times \mathbb{R}^+} \widehat{f}(\xi) \widehat{\psi}(t(\xi - c) - \eta')^2 \chi(\eta') d\eta' \frac{dt}{t} \\
&= \widehat{f}(\xi) \int_{B_\varepsilon(d)} \mathbb{1}_{[c, +\infty)}(\xi) C_{\psi, \eta'} \chi(\eta') d\eta' = \widehat{f}(\xi) \mathbb{1}_{[c, \infty)}(\xi)
\end{aligned}$$

where  $C_{\psi, \eta'} = \int_{\mathbb{R}^+} \widehat{\psi}(t - \eta')^2 \frac{dt}{t}$ . The last identity holds up to renormalizing  $\psi$ . Clearly, notice that if  $\eta' \in B_\varepsilon(d)$  and  $\xi - c < 0$  then  $t(\xi - c) - \eta' < -b$  for all  $t \in \mathbb{R}^+$  and thus  $\widehat{\psi}(t(\xi - c) - \eta')^2 = 0$ . On the other hand if  $\xi - c > 0$  one may change variables  $t' = t(\xi - c)$  to obtain

$$\int_{\mathbb{R}^+} \widehat{\psi}(t(\xi - c) - \eta')^2 \frac{dt}{t} = \int_{\mathbb{R}^+} \widehat{\psi}(t' - \eta')^2 \frac{dt'}{t'} = C_{\psi, \eta'}$$

The constant is positive and finite since  $t' \mapsto \widehat{\psi}(t' - \eta')$  is real-valued and supported away from 0. It can be checked that the integral in (1.36) converges uniformly.  $\square$

We claim that the  $W_2$  Walsh analog with of the wave packet representation (0.3) is given by

$$\sum_{P \in \mathbb{P}} \langle f; w_P \rangle \langle a(\cdot); \mathbb{1}_{\omega_{P_d}}(c(\cdot)) w_P(\cdot) \rangle |I_P| = \sum_{P \in \mathbb{P}} \mathbb{F}(P) \mathbb{A}(P) |I_P| \quad (1.40)$$

where  $\omega_{P_d}$  is the lower dyadic sibling of  $\omega_P$  i.e. if  $\omega'$  is the dyadic sibling of  $\omega_P$  then  $\omega_{P_d} = \omega'$  if  $\omega'$  is below  $\omega_P$  and  $\omega_{P_d} = \emptyset$  otherwise.

The integral in (0.3) is taken over  $\mathbb{R}_+^3$  that parametrizes all rectangles  $I_P \times \omega_P$  of  $\mathbb{R}^2$  with area 1 via the map

$$(y, \eta, t) \mapsto B_t(y) \times B_{t^{-1}}(\eta).$$

The term  $|I_P|$  in (1.40) is simply a normalization term associated to the discretization of the measure of  $\mathbb{R}_+^3$  when passing to dyadic tiles.

A real wave packet

$$\varphi_{\eta,t}(\cdot - z) = e^{i\eta(x-z)} t^{-1} \varphi\left(\frac{\cdot - z}{t}\right)$$

is adapted to such a tile in the sense that  $\varphi_{\eta,t}(\cdot - z)$  is essentially supported on  $B_t(y)$  whereas  $\widehat{\varphi_{\eta,t}(\cdot - z)}$  is essentially supported on  $B_{t^{-1}}(\eta)$ . By this we mean that for arbitrary  $N > 0$  the bounds

$$\begin{aligned}
|\varphi_{\eta,t}(\cdot - y)| &\lesssim C_N t^{-1} \left(1 + \frac{|\cdot - y|}{t}\right)^{-N} \\
|\widehat{\varphi_{\eta,t}(\cdot - y)}(\xi)| &\lesssim C_N t \left(1 + t|\cdot - \eta|\right)^{-N}
\end{aligned}$$

hold. The Walsh analog for  $\varphi$  is given by  $\mathbb{1}_{B_1(0)}$  so that  $\varphi_{\eta,t}(\cdot - y)$  and  $w_P(\cdot)$  have corresponding time-frequency localization for  $t = |I_P|$ ,  $y = x_P$ , and  $\eta = \eta_P$  and thus the Walsh embedding

$$\mathbb{F}(P) = \langle f; w_P \rangle = \int_{W_P} f(x) \overline{w_P(x)} dx \quad (1.41)$$

corresponds to (0.5). On the other hand the truncated wave packet (1.39) either essentially coincides with  $\psi_{\eta,t}$  if

$$B_{t^{-1}}(\eta) \subset (c, +\infty] \quad \text{dist}(c; B_{t^{-1}}(\eta)) \approx t^{-1}$$

or vanishes if the above conditions do not hold. It follows that

$$\mathbb{A}(P) = \langle a(\cdot); \mathbb{1}_{\omega_{P_d}}(c(\cdot))w_P(\cdot) \rangle \quad (1.42)$$

is the appropriate Walsh analog for the embedding (0.6). Given the definition of  $\omega_{P_d}$  we can restrict  $\mathbb{P}$  to consist only of the tiles  $P$  that are the higher frequency children of their parent i.e.  $\omega_{P_d} \neq \emptyset$ .

### 1.3.3 Boundedness of the Walsh Carleson operator in local $L^2$

Using the theory introduced in Section 1.1 we will now prove that the bounds

$$\left| \sum_{P \in \mathbb{P}} \mathbb{F}(P)\mathbb{A}(P)|I_P| \right| \lesssim \|f\|_{L^p} \|a\|_{L^{p'}} \quad p \in (2, \infty) \quad (1.43)$$

hold.

We begin by defining the family of generating sets  $\mathbb{T}$  that we call trees. For every tile  $P_T = I_T \times \omega_T \subset \mathbb{P}$  set

$$\begin{aligned} T(P_T) &:= T^{(d)}(T_P) \cup T^{(u)}(T_P) := \\ &\{P \in \mathbb{P}: I_P \subseteq I_T, \omega_T \subseteq \omega_P\} \cup \{P \in \mathbb{P}: I_P \subseteq I_T, \omega_T \subseteq \omega_{P_d}\} \end{aligned} \quad (1.44)$$

where  $\omega_{P_d}$  is the lower dyadic sibling of  $\omega_P$  as defined previously. Let us define a pre-measure by setting

$$\bar{\mu}(T) = |I_T|. \quad (1.45)$$

and let the outer measure be generated by  $\bar{\mu}$  as described in (1.1). Consider the two generating sizes

$$\begin{aligned} \|\mathbb{F}\|_{S^e(T)} &:= \|\mathbb{F}\|_{S^{2,u}(T)} + \|\mathbb{F}\|_{S^\infty(T)} \\ &:= \left( \frac{1}{|I_T|} \sum_{P \in T^{(u)}} |\mathbb{F}(P)|^2 |I_P| \right)^{1/2} + \sup_{P \in T} |\mathbb{F}(P)| \end{aligned} \quad (1.46)$$

$$\begin{aligned} \|\mathbb{A}\|_{S^m(T)} &:= \|\mathbb{A}\|_{S^{2,u}(T)} + \|\mathbb{A}\|_{S^{1,d}(T)} \\ &:= \left( \frac{1}{|I_T|} \sum_{P \in T^{(u)}} |\mathbb{A}(P)|^2 |I_P| \right)^{1/2} + \frac{1}{|I_T|} \sum_{P \in T^{(d)}} |\mathbb{A}(P)| |I_P|. \end{aligned} \quad (1.47)$$

that are Hölder dual with respect to the size  $S^1$  generated by

$$\|\mathbb{G}\|_{S^1(T)} := \frac{1}{|I_T|} \sum_{P \in T} |\mathbb{G}(P)| |I_P| \quad (1.48)$$

in the sense that

$$\|\mathbb{F}\mathbb{A}\|_{S^1} \lesssim \|\mathbb{F}\|_{S^e} \|\mathbb{A}\|_{S^m} \quad \forall \mathbb{F}, \mathbb{A} \text{ Borel functions.} \quad (1.49)$$

Using the outer integral domination property 1.10 outer-measure Hölder inequality 1.6 it follows that

$$\left| \sum_{P \in \mathbb{P}} \mathbb{F}(P)\mathbb{A}(P)|I_P| \right| \lesssim \|\mathbb{F}\mathbb{A}\|_{S^1} \lesssim \|\mathbb{F}\|_{L^p S^e} \|\mathbb{A}\|_{L^{p'} S^m} \quad p \in [1, \infty] \quad (1.50)$$



To show (1.43) it is thus sufficient to show the embedding bounds

$$\|\mathbb{F}\|_{L^p S^e} \lesssim \|f\|_{L^p} \quad p \in (2, \infty] \quad (1.51)$$

$$\|\mathbb{A}\|_{L^{p'} S^m} \lesssim \|a\|_{L^{p'}} \quad p' \in (1, \infty] \quad (1.52)$$

### 1.3.4 Proof of bounds for $\mathbb{F}$

We show that (1.51) holds with  $p = \infty$  and that the weak bound

$$\|F\|_{L^{2,\infty} S^e} \lesssim \|f\|_{L^2} \quad (1.53)$$

holds. The statement then follows by outer-measure interpolation as in Proposition 1.7.

For  $p = \infty$  we show that for any tree  $T$  the bound

$$\|F\|_{S^e(T)} = \|F\|_{S^{2,u}(T)} + \|F\|_{S^\infty(T)} \lesssim \|f\|_{L^\infty}$$

holds. Since  $\|w_P\|_{L^1} = 1$  it follows that

$$\|F\|_{S^\infty(T)} = \sup_{P \in T} \int_{W_2} f \overline{w_P} \leq \|f\|_{L^\infty}.$$

It remains to show that

$$\|F\|_{S^{2,u}(T)}^2 = \frac{1}{|I_T|} \sum_{P \in T^{(u)}} |\mathbb{F}(P)|^2 |I_P| \lesssim \|f\|_{L^2}^2.$$

Notice that the tiles  $P \in T^{(u)}$  are pairwise disjoint. Given two distinct tile  $P, P' \in T^{(u)}$  it holds that  $\omega_T \subseteq \omega_{P_d} \cap \omega_{P'_d} \neq \emptyset$  so let us suppose that  $\omega_{P_d} \subseteq \omega_{P'_d}$ . If  $\omega_{P_d} = \omega_{P'_d}$  and  $I_P \cap I_{P'} \neq \emptyset$  then  $I_P = I_{P'}$  since  $|I_P| = |\omega_{P_d}|^{-1} = |\omega_{P'_d}|^{-1} = |I_{P'}|$  and thus  $P, P'$  would coincide. On the other hand, if  $\omega_{P_d} \subsetneq \omega_{P'_d}$  then  $\omega_P \cap \omega_{P'} = \emptyset$ . Thus  $|I_P|^{1/2} w_P$  are orthonormal for  $P \in T^{(u)}$  and are all supported on  $I_T$  so by Bessel's inequality

$$\sum_{P \in T^{(u)}} \mathbb{F}(P)^2 |I_P| = \sum_{P \in T^{(u)}} \langle f; w_P \rangle^2 |I_P| \leq \|f \mathbb{1}_{I_T}\|_{L^2}^2 \lesssim |I_T| \|f\|_{L^\infty}^2$$

as required.

Proving the bounds (1.53) requires showing that for any  $\lambda > 0$  there exists a set  $E_\lambda \subset \mathbb{P}$  such that

$$\|\mathbb{F} \mathbb{1}_{E_\lambda}\|_{S^e(T)} \lesssim \lambda \quad \forall T \in \mathbb{T}, \quad \mu(E_\lambda) \leq \frac{\|f\|_{L^2}^2}{\lambda^2} \quad (1.54)$$

It is enough to show that the above holds for  $S^\infty$  and  $S^{2,u}$  in lieu of  $S^e$  respectively.

Let us first work with  $S^{2,u}$ ; we construct  $E_\lambda$  via an algorithmic covering argument. At every step  $j \geq 1$ , among the  $T$  such that

$$\frac{1}{|I_T|} \sum_{P \in T^{(u)} \setminus \bigcup_{k=1}^{j-1} T_k} |\mathbb{F}(P)|^2 |I_P| \geq \lambda^2 \quad (1.55)$$

consider those trees  $T$  with minimal  $\omega_T$  with respect to inclusion. This is possible since for any tree it holds that

$$|I_T| \leq \lambda^{-2} \sum_{P \in T^{(u)}} |\mathbb{F}(P)|^2 |I_P|$$

and thus  $|\omega_T|$  is bounded from below. Consider the tree  $T_j$  to be the one with the higher-most  $\omega_T$  among this sub-collection .

We claim that

$$P \in (T_j^{(u)} \setminus \bigcup_{k=1}^{j-1} T_k), P' \in (T_{j'}^{(u)} \setminus \bigcup_{k=1}^{j'-1} T_k) \implies P \cap P' = \emptyset \quad (1.56)$$

for any  $j \leq j'$ . If  $j = j'$  then  $P, P' \in T_j^{(u)}$  and we reduce to the single tree case that we have shown previously. For  $j < j'$  let us reason by contradiction and suppose that  $\omega_P \cap \omega_{P'} \neq \emptyset$  and  $I_P \cap I_{P'} \neq \emptyset$ . If  $\omega_P = \omega_{P'}$  then it follows that  $P = P'$  and this contradicts the assumptions of (1.56) since  $P' = P \in T_j^{(u)} \subset \bigcup_{k=1}^{j'-1} T_k$ . Suppose that  $\omega_P \subsetneq \omega_{P'}$  then  $\omega_{T_j} \subset \omega_{P_d} \subset \omega_{P'}$  and  $I_{P'} \subset I_P \subset I_{T_j}$  and thus  $P' \in T_j^{(u)}$  once again contradicting the assumptions of (1.56). Finally suppose  $\omega_{P'} \subsetneq \omega_P$  then  $\omega_{T_{j'}} \subset \omega_{P'_d} \subset \omega_P$  while  $\omega_T \subset \omega_{P_d}$  so  $\omega_{T_{j'}}$  is higher than  $\omega_{T_j}$  that contradicts the selection order.

Disjointness property (1.56) gives that, at any step  $N$ , one has

$$\begin{aligned} \|f\|_{L^2}^2 &\geq \sum_{j=0}^N \sum_{P \in T_j^{(u)} \setminus \bigcup_{k=1}^{j-1} T_k} \langle f; w_P \rangle^2 |I_P| \\ &= \sum_{j=1}^N \sum_{P \in T_j^{(u)} \setminus \bigcup_{k=1}^{j-1} T_k} |\mathbb{F}(P)|^2 |I_P| \geq \sum_{j=1}^N \lambda^2 |I_{T_j}|. \end{aligned}$$

We used Bessel's inequality since disjointness implies orthogonality according to Proposition 1.13. It follows that

$$\mu\left(\bigcup_{j=1}^N T_j\right) \leq \sum_{j=1}^N |I_{T_j}| \leq \frac{\|f\|_{L^2}^2}{\lambda^2}$$

and thus  $E_\lambda = \bigcup_{j=1}^\infty T_j$  also satisfies  $\mu(E_\lambda) \leq \frac{\|f\|_{L^2}^2}{\lambda^2}$ . Conditions (1.54) with  $S^{2,(u)}$  in lieu of  $S^e$  are satisfied: in fact any tree  $T$  where the first condition fails would be selected by the above algorithm in finitely many steps.

To show that (1.54) holds with  $S^\infty$  in lieu of  $S^e$  one may follow a similar algorithm using

$$\frac{|\mathbb{F}(P_T)| \mathbb{1}_{\mathbb{P} \setminus \bigcup_{k=1}^j T_k}(P_T)}{|I_T|} \geq \lambda$$

as the selection condition instead of (1.55). We omit the details that are essentially the same.

### 1.3.5 Proof of bounds for $\mathbb{A}$

We show that bounds (1.52) hold by interpolating between  $p' = \infty$  and the weak  $L^{p'}$  bounds that hold for  $p' = 1$ .

Let us associate to every tile  $P \in \mathbb{P}$  the auxiliary embedding

$$\mathbb{M}(P) = \frac{1}{|I_P|} \int_{I_P} |a(x)| \mathbb{1}_{\omega_P}(c(x)) dx \quad (1.57)$$

and we show that the bounds

$$\begin{aligned} \|\mathbb{M}\|_{L^{p'}, S^\infty} &\lesssim \|a\|_{L^{p'}} \quad p' \in (1, \infty) \\ \|\mathbb{M}\|_{L^{1, \infty} S^\infty} &\lesssim \|a\|_{L^1} \end{aligned} \quad (1.58)$$

hold with  $\|\mathbb{M}\|_{S^\infty} = \sup_{P \in \mathbb{P}} |\mathbb{M}(P)|$ . We then conclude using the following size domination lemma

**Lemma 1.14.** *Suppose  $E = \bigcup_n T_n$  is a union of tree such that*

$$\|\mathbb{M}(P)\mathbb{1}_{\mathbb{P} \setminus E}\|_{S^\infty} < \lambda$$

then

$$\|\mathbb{A}\mathbb{1}_{\mathbb{P} \setminus E}\|_{S^m} \lesssim \lambda.$$

*Proof of (1.58).* We show (1.58) by interpolating between the two endpoint estimates. For  $p' = \infty$  the estimates is trivial since

$$\mathbb{M}(P) = \frac{1}{|I_P|} \int_{I_P} |a(x)| \mathbb{1}_{\omega_P}(c(x)) dx \leq \|a\|_{L^\infty}.$$

We now show the weak  $L^1 S^\infty$  bounds: for any  $\lambda > 0$  we find a set  $E_\lambda$  such that

$$\|\mathbb{M}\mathbb{1}_{\mathbb{P} \setminus E_\lambda}\|_{S^\infty} \lesssim \lambda \quad \mu(E_\lambda) \leq \lambda^{-1} \|a\|_{L^1} \quad (1.59)$$

We proceed algorithmically. At every step  $j$  consider let  $P_j$  be a tile with minimal  $\omega_P$  with respect to inclusion among tiles satisfying

$$P \in \mathbb{P} \setminus \bigcup_{k=1}^{j-1} T(P) \quad |\mathbb{M}(P)| \geq \lambda. \quad (1.60)$$

This is always possible since for any such  $P$  it holds that  $|I_P| < \lambda^{-1} \|a\|_{L^1}$ .

We claim that

$$P_j \cap P_{j'} = \emptyset \quad \text{for any } j < j'. \quad (1.61)$$

Let us reason by contradiction and suppose that  $\omega_{P_j} \cap \omega_{P_{j'}} \neq \emptyset$  and  $I_{P_j} \cap I_{P_{j'}} \neq \emptyset$ . By minimality with respect to inclusion  $\omega_{P_j} \subset \omega_{P_{j'}}$ , but then  $P_{j'} \in T(P_j)$  contradicting (1.60).

Using (1.61), at any step  $N$ , we obtain the bound

$$\begin{aligned} \mu\left(\bigcup_{j=1}^N T(P_j)\right) &\leq \sum_{j=1}^N |I_{P_j}| = \sum_{j=1}^N |I_P| \frac{|\mathbb{M}(P_j)|}{\lambda} \\ &\leq \lambda^{-1} \int_{W_2} |a(x)| \sum_{j=1}^N \mathbb{1}_{I_{P_j}}(x) \mathbb{1}_{\omega_P}(c(x)) dx \leq \frac{\|a\|_{L^1}}{\lambda}. \end{aligned}$$

Setting  $E_\lambda = \bigcup_{j=1}^\infty T(P_j)$  we obtain (1.59). In fact any  $P$  that violates would be selected after finitely many steps.  $\square$

*Proof of size domination Lemma 1.14.* Given  $E = \bigcup_n T_n$  such that  $\|\mathbb{M}\mathbb{1}_{\mathbb{P} \setminus E}\|_{S^\infty} \leq \lambda$  we show that for any tree  $T$  the bounds

$$\|\mathbb{A}\mathbb{1}_{\mathbb{P} \setminus E}\|_{S^{1,(d)}(T)} \lesssim \lambda \quad (1.62)$$

$$\|\mathbb{A}\mathbb{1}_{\mathbb{P} \setminus E}\|_{S^{2,(u)}(T)} \lesssim \lambda \quad (1.63)$$

hold.

Let us begin with (1.62) by showing that for any  $N > 0$  the following bound holds

$$\sum_{\Delta T^{(d)}} |\mathbb{A}(P)| |I_P| \lesssim \lambda |I_T| \quad \text{where } \Delta T^{(d)} = (T^{(d)} \setminus E) \cap \{P: |I_P| > 2^{-N}\}.$$

Let  $\mathcal{J}$  be the partition of  $I_T$  generated by  $I_P$  with  $P \in \Delta T^{(d)}$  i.e. setting

$$\mathcal{I} = \{I \subset I_T : I_P \not\subseteq I \forall P \in \Delta T^{(d)}\}$$

$J \in \mathcal{J}$  if and only if  $J \in \mathcal{I}$  and  $2J \notin \mathcal{I}$ .

Then

$$\begin{aligned} \sum_{\Delta T^{(d)}} |\mathbb{A}(P)| |I_P| &\leq \sum_{J \in \mathcal{J}} \int_J |a(x)| \sum_{P \in \Delta T^{(d)}} \mathbb{1}_{I_P}(x) \mathbb{1}_{\omega_{P_d}}(c(x)) \, dx \\ &\leq \sum_{J \in \mathcal{J}} |J| \int_J |a(x)| \mathbb{1}_{\omega_J}(c(x)) \, dx \end{aligned}$$

where  $\omega_J$  is the unique dyadic interval such that  $\omega_T \subset \omega_J$  and  $|\omega_J| = |J|^{-1}$ . This holds since if  $x \in J \cap I_P$  for some  $P \in \Delta T^{(d)}$  then  $J \not\subseteq I_P$  and thus  $\omega_{P_d} \subset 2\omega_P \subseteq \omega_J$ . It is sufficient to show that

$$\int_J |a(x)| \mathbb{1}_{\omega_J}(c(x)) \leq 4\lambda. \quad (1.64)$$

Reasoning by contradiction let  $\omega_J^+$  and  $\omega_J^-$  be the two (upper and lower) dyadic children of  $\omega_J$  i.e.  $2\omega_J^+ = 2\omega_J^- = \omega_J$  and set  $Q^+ = 2J \times \omega_J^+$  and  $Q^- = 2J \times \omega_J^-$  so that at

$$\mathbb{M}(Q^+) \geq \lambda \text{ or } \mathbb{M}(Q^-) \geq \lambda$$

and thus  $Q^+ \in E$  or  $Q^- \in E$ . By maximality of  $J$  there exists  $P \in \Delta T^{(d)}$  such that  $I_P \subseteq 2J$  but then  $P \in E$  since either  $\omega_J^+ \subset \omega_P$  or  $\omega_J^- \subset \omega_{P_d}$  and  $E$  is a union of trees. This concludes the proof for  $S^{1,(d)}$ .

We show (1.63) by proving for any tree  $T$  and any  $N > 0$  the dual bounds

$$\sum_{P \in \Delta T^{(u)}} \mathbb{A}(P) \mathbb{H}(P) |I_P| \lesssim \lambda |I_T| \quad \text{where } \Delta T^{(u)} = (T^{(u)} \setminus E) \cap \{P : |I_P| > 2^{-N}\}. \quad (1.65)$$

for any function  $\mathbb{H}$  with

$$\sum_{P \in \Delta T^{(u)}} |\mathbb{H}(P)|^2 |I_P| \leq |I_T|.$$

Let once again  $\mathcal{J}$  be the partition generated by  $P \in \Delta T^{(u)}$ .

We can rewrite the sum in (1.65) as

$$\begin{aligned} \sum_{P \in \Delta T^{(u)}} \mathbb{A}(P) \mathbb{H}(P) |I_P| &= \int_{I_T} a(x) \sum_{P \in \Delta T^{(u)}} \mathbb{1}_{\omega_{P_d}}(c(x)) \mathbb{H}(P) w_P(x) |I_P| \, dx \\ &= \sum_{J \in \mathcal{J}} \int_J a(x) \mathbb{1}_J(x) \mathbb{1}_{\omega_J}(x) \sum_{P \in \Delta T^{(u)}} \mathbb{1}_{\omega_{P_d}}(c(x)) \mathbb{H}(P) w_P(x) |I_P| \, dx. \end{aligned}$$

The second equality holds by maximality of  $J$  since if  $x \in J \not\subseteq I_P$  then  $\omega_J \supset \omega_{P_d}$ . Let

$$h(x) = \sum_{P \in \Delta T^{(u)}} \mathbb{H}(P) w_P(x) |I_P|$$

and since tiles in  $\Delta T^{(u)}$  are pairwise disjoint and thus  $w_P$  are orthogonal we have that

$$\|h\|_{L^2}^2 = \sum_{P \in \Delta T^{(u)}} |\mathbb{H}(P)|^2 |I_P| \leq |I_T|.$$

For any given  $x \in I_T$ , set

$$\begin{aligned} l_-(x) &= \min\{|I_P| \text{ with } P \in \Delta T^{(u)}, c(x) \in \omega_{P_d}\} \\ l_+(x) &= \max\{|I_P| \text{ with } P \in \Delta T^{(u)}, c(x) \in \omega_{P_d}\} \end{aligned}$$

and let  $P_-(x)$  be a tile such that  $|I_{P_-(x)}| = \frac{l_-(x)}{2}$  and  $\omega_T \subset \omega_{P_-(x)}$  and let  $P_+(x)$  be such that  $|I_{P_+(x)}| = l_+(x)$  and  $\omega_T \subset \omega_{P_+(x)}$ . Notice that for every point  $x \in I_T$  it holds that

$$\begin{aligned} \sum_{P \in \Delta T^{(u)}} \mathbb{1}_{\omega_{P_d}}(c(x)) \mathbb{H}(P) w_P(x) |I_P| &= \left( \int h(y) w_{P_-(x)}(y) dy \right) w_{P_-(x)}(x) |I_{P_-(x)}| \\ &\quad - \left( \int h(y) w_{P_+(x)}(y) dy \right) w_{P_+(x)}(x) |I_{P_+(x)}| \end{aligned}$$

and thus

$$\left| \sum_{P \in \Delta T^{(u)}} \mathbb{1}_{\omega_{P_d}}(c(x)) \mathbb{H}(P) w_P(x) |I_P| \right| \lesssim \sup_{l > l_-(x)} \int_{B_l(x)} |h(y)| dy.$$

Using the fact that  $l_-(x) > |J|$  for any  $x \in J \in \mathcal{J}$  the right We have that for  $x \in J$

$$\sup_{l > l_-(x)} \int_{B_l(x)} |h(y)| dy \leq \sup_{I \supset J} \int_I |h(y)| dy$$

where the right hand side is constant for  $x \in J$  and is dominated by the dyadic Hardy-Littlewood maximal function

$$Mh(x) = \sup_{I \supset x} \int_I |h(x)| dx.$$

Thus

$$\begin{aligned} \left| \int_J a(x) \mathbb{1}_J(x) \mathbb{1}_{\omega_J}(x) \sum_{P \in \Delta T^{(u)}} \mathbb{1}_{\omega_{P_d}}(c(x)) \mathbb{H}(P) w_P(x) |I_P| dx \right| \\ \lesssim \int_J a(x) \mathbb{1}_J(x) \mathbb{1}_{\omega_J}(x) dx \int_J Mh(x) dx. \end{aligned}$$

and it can be shown as in the case (1.64) that

$$\int_J a(x) \mathbb{1}_J(x) \mathbb{1}_{\omega_J}(x) dx \leq 4\lambda.$$

Using the  $L^2$  bound on the dyadic Hardy-Littlewood maximal function we conclude that

$$\left| \sum_{P \in \Delta T^{(u)}} \mathbb{A}(P) \mathbb{H}(P) |I_P| \right| \lesssim \lambda \int_{I_T} Mh(x) dx \lesssim \lambda |I_T|$$

as required. □

## 1.4 Iterated outer measure spaces

In the previous section we have shown that the Walsh analog of the Carleson operator (0.1) is bounded on  $L^p(\mathbb{R})$  for  $p \in (2, \infty)$  by reducing to the bounds (1.51) and (1.52) on two embedding

maps. However the bounds (1.51) do not hold for  $p \in (1, 2)$  and this is the main obstruction for proving bounds for the Carleson operator on the full range  $p \in (1, \infty)$ .

The failure of (1.51) for  $p < 2$  follows from a rescaling argument. Consider the function  $f = \mathbb{1}_{B_1(0)}$  and consider the quantity  $\mu(\|\mathbb{F}\|_{S^e} > 2^{-k})$  for  $k > 0$  large enough. Notice that

$$\mathbb{F}(P) \geq 2^{-k} \quad \forall P = B_{2^k}(0) \times B_{2^{-k}}(\xi) \text{ with } B_{2^{-k}}(\xi) \subset B_1(0)$$

thus if  $\|\mathbb{F}\mathbb{1}_{\mathbb{X} \setminus E}\|_{S^e} < 2^{-k}$  then  $E$  has to contain all the above tiles. Any given tree  $T \in \mathbb{T}$  can contain at most two such tiles and each such tree has to have  $|I_T| \geq 2^k$ . Since there are  $2^k$  tiles as above it follows that

$$\mu(\|\mathbb{F}\|_{S^e} > 2^{-k}) \geq 2^{2k-1}$$

so clearly even weak bounds fail for  $p < 2$ .

Intuitively, the problem is that the measure  $\mu$  does not distinguish space and frequency localization: in the above example the “optimal” covering set  $\|\mathbb{F}\mathbb{1}_{\mathbb{X} \setminus E}\|_{S^e} < 2^{-k}$  consists of trees at different top frequencies  $\xi_T$  but with the same spatial interval  $I_T = B_{2^k}(0)$ .

We address this deficiency by introducing iterated outer measure spaces. Consider a new family of generating sets  $\mathbb{D}$  called strips given by

$$D(I_D) = \{P \in \mathbb{P} : I_P \subset I_D\}$$

and let us generate an outer measure  $\nu$  via the pre-measure

$$\bar{\nu}(D) = |I_D|.$$

As sizes we use localized versions of outer-measure  $L^p$  norms calculated with respect to the outer measure  $\mu$ . For every  $q \in (0, \infty]$  and for every strip  $D \in \mathbb{D}$  introduce the size

$$\|\mathbb{F}\|_{\mathcal{L}^q(D)S} = |I_D|^{-1/q} \|\mathbb{F}\mathbb{1}_D\|_{L^q S} \quad \|\mathbb{F}\|_{\mathcal{L}^q S} = \sup_{D \in \mathbb{D}} \|\mathbb{F}\|_{\mathcal{L}^q(D)S} \quad (1.66)$$

It is straight-forward to check that  $\mathcal{L}^q S$  satisfy the conditions for being a size. We can thus define iterated outer-measure sizes as follows

$$\|\mathbb{F}\|_{L^p_\nu \mathcal{L}^q_\mu S} = \int_{\mathbb{R}^+} p\lambda^p \nu(\|\mathbb{F}\|_{\mathcal{L}^q_\mu S} > \lambda) \frac{d\lambda}{\lambda}.$$

In this instance we explicitly specify the outer measure of the iterated outer-measure  $L^p$  spaces. We claim that the following bounds hold for the two embeddings (1.41) and (1.42):

$$\|\mathbb{F}\|_{L^p \mathcal{L}^q S^e} \lesssim_{p,q} \|f\|_{L^p} \quad p \in (1, \infty], q \in (\max(2, p'), \infty] \quad (1.67)$$

$$\|\mathbb{A}\|_{L^{p'} \mathcal{L}^{q'} S^m} \lesssim_{p,q} \|f\|_{L^p} \quad p \in (1, \infty], q \in (1, \infty]. \quad (1.68)$$

Before proceeding to the proof of these two statements let us see how this allows us to deduce  $L^p$  bounds for the form (1.40) associated to the Walsh Carleson operator on  $L^p \times L^{p'}$  for the full range  $p \in (1, \infty)$ .

As a matter of fact it can be shown that

$$\left| \sum_{P \in \mathbb{P}} \mathbb{F}(P) \mathbb{A}(P) |I_P| \right| \lesssim \|\mathbb{F} \mathbb{A}\|_{L^1_\nu \mathcal{L}^1_\mu S^1}$$

by using the property 1.10 twice. To apply it at the outer level of  $\nu$  and  $\mathcal{L}^1_\mu S^1$  it is sufficient to verify that for any  $D \in \mathbb{D}$

$$\left| \sum_{P \in D} \mathbb{G}(P) |I_P| \right| \lesssim \bar{\nu}(D) \|\mathbb{G}\|_{\mathcal{L}^1_\mu(D)S^1}$$

But this follows once again by property 1.10 recalling the definition of the size (1.48) and of the measure (1.45) on trees. It holds that

$$\left| \sum_{P \in D} \mathbb{G}(P) |I_P| \right| \lesssim \|\mathbb{G} \mathbb{1}_D\|_{L^1_\mu S^1}$$

since for every tree  $T \in \mathbb{T}$

$$\left| \sum_{P \in T} \mathbb{G}(P) \mathbb{1}_D |I_P| \right| \lesssim \mu(T) \|\mathbb{G} \mathbb{1}_D\|_{S^1(T)}.$$

Similarly, using outer-measure Hölder's inequality 1.6 twice, it follows that

$$\|\mathbb{F} \mathbb{A}\|_{L^1_\nu \mathbb{E}^1_\mu S^1} \lesssim \|\mathbb{F}\|_{L^p_\nu \mathbb{E}^q_\mu S^e} \|\mathbb{A}\|_{L^{p'}_{\nu'} \mathbb{E}^{q'}_{\mu'} S^m} \quad \frac{1}{p} + \frac{1}{p'} = 1 \quad \frac{1}{q} + \frac{1}{q'} = 1$$

Clearly the above bounds hold as long as

$$\|\mathbb{F} \mathbb{A}\|_{\mathbb{E}^1_\mu S^1} \lesssim \|\mathbb{F}\|_{\mathbb{E}^q_\mu S^e} \|\mathbb{A}\|_{\mathbb{E}^{q'}_{\mu'} S^m}.$$

but this in turn follows from outer-measure Hölder's inequality and the duality relation (1.49). Putting the above steps together gives

$$\left| \sum_{P \in \mathbb{P}} \mathbb{F}(P) \mathbb{A}(P) |I_P| \right| \lesssim \|\mathbb{F} \mathbb{A}\|_{L^1_\nu \mathbb{E}^1_\mu S^1} \lesssim \|\mathbb{F}\|_{L^p_\nu \mathbb{E}^q_\mu S^e} \|\mathbb{A}\|_{L^{p'}_{\nu'} \mathbb{E}^{q'}_{\mu'} S^m} \lesssim \|f\|_{L^p} \|a\|_{L^{p'}}$$

for all  $p \in (1, \infty)$  since for each such  $p$  one can find a Hölder tuple  $(q, q')$  such that bounds (1.67) and (1.68) hold.

We now pass to proving bounds (1.67) and (1.68).

### 1.4.1 Proof of iterated bounds for $\mathbb{F}$

We will show bounds (1.67) by interpolating between weak endpoint results. In particular we show that  $p \in (1, 2)$  the two bounds

$$\|\mathbb{F}\|_{L^{p, \infty}_\nu \mathbb{E}^\infty_\mu S^e} \lesssim \|f\|_{L^p} \tag{1.69}$$

$$\|\mathbb{F}\|_{L^{p, \infty}_\nu \mathbb{E}^{p', \infty}_{\mu'} S^e} \lesssim \|f\|_{L^p} \tag{1.70}$$

hold.

We begin with (1.69) and we need to show that for any  $\lambda > 0$  there exists a set  $K_\lambda \subset \mathcal{P}$  such that

$$\nu(K_\lambda) \lesssim \lambda^{-p} \|f\|_{L^p}^p \quad \|\mathbb{F} \mathbb{1}_{\mathbb{P} \setminus K_\lambda}\|_{\mathbb{E}^\infty S^e} \lesssim \lambda. \tag{1.71}$$

Let us consider the dyadic Hardy-Littlewood  $p$  maximal function

$$M_p f(x) = \sup_{I \ni x} \left( \int_I |f(x)|^p dx \right)^{1/p}$$

where the supremum is taken over all dyadic intervals. Since  $M_p$  is bounded on  $L^p$  it holds that

$$\begin{aligned} \{x : M_p f(x) > \lambda\} &= \bigcup_{n \in \mathbb{N}} I_n & K_\lambda &= \bigcup_{n \in \mathbb{N}} D_n = \bigcup_{n \in \mathbb{N}} D(I_n) \\ \nu(K_\lambda) &\lesssim \sum_{n \in \mathbb{N}} |I_n| & &\lesssim \lambda^{-p} \|f\|_{L^p}^p \end{aligned} \tag{1.72}$$

where  $I_n$  are the maximal dyadic intervals with respect to inclusion contained in  $\{x: M_1 f(x) > \lambda\}$ . It remains to show the second claim of (1.71) i.e. that for any tree  $T \in \mathbb{T}$  it holds that

$$\|\mathbb{F}\mathbb{1}_{\mathbb{P}\setminus K_\lambda}\|_{S^e(T)} \lesssim \lambda. \quad (1.73)$$

If  $I_T \subset I_n$  for some  $n \in \mathbb{N}$  then  $\mathbb{F}\mathbb{1}_{\mathbb{P}\setminus K_\lambda} = 0$  and there is nothing to prove. Otherwise let us do a Calderón-Zygmund decomposition of  $f\mathbb{1}_{I_T}$  at frequency  $\xi_T \in \omega_T$  over the intervals  $I_n$  from (1.72) for which  $I_n \subset I_T$ .

By this decomposition we mean that

$$f\mathbb{1}_{I_T} = g + \sum_{n \in \mathbb{N}} b_n$$

where

$$g := g_c + \sum_{n \in \mathbb{N}} g_n \quad g_c = f\mathbb{1}_{I_T \setminus \bigcup_{n \in \mathbb{N}} I_n} \quad g_n = \langle f; w_{P_n} \rangle w_{P_n} |I_n|$$

where  $P_n = I_n \times B_{|I_n|^{-1}}(\xi_T)$  is the unique tile  $P \in \mathbb{P}$  such that  $I_P = I_n$  and  $\omega_T \subset \omega_P$ . The “bad” terms are given by

$$b_n = f\mathbb{1}_{I_T} \mathbb{1}_{I_n} - g_n$$

and notice that for any  $P \in T \setminus \bigcup_{n \in \mathbb{N}} D_n$  it holds that

$$\langle b_n; w_P \rangle = 0.$$

and thus

$$\mathbb{F}(P)\mathbb{1}_{\mathbb{P}\setminus K_\lambda}(P) = \langle g; w_P \rangle \mathbb{1}_{\mathbb{P}\setminus K_\lambda}(P).$$

Notice that for every  $x \in W_2$  it holds that  $|g_c(x)| \leq \lambda$  and  $|g_n(x)| \lesssim \lambda$  by maximality of  $I_n$ . Thus  $|g(x)| \lesssim \lambda$  and (1.73) follows by  $L^\infty$  boundedness of the embedding (1.41). We have concluded the proof of (1.69).

To show bounds (1.70) we must prove that for every  $\lambda > 0$  there exists a set  $K_\lambda$  such that

$$\nu(K_\lambda) \lesssim \lambda^{-p} \|f\|_{L^p} \quad \|\mathbb{F}\mathbb{1}_{\mathbb{P}\setminus K_\lambda}\|_{\mathcal{E}^{p', \infty} S^e} \lesssim \lambda. \quad (1.74)$$

The second condition means that for any  $D \in \mathbb{D}$  and for any  $\tau > 0$  there exists  $E_\tau$  depending on  $D$  and  $\tau$  such that

$$\|\mathbb{F}\mathbb{1}_{D \setminus K_\lambda} \mathbb{1}_{\mathbb{P}\setminus E_\tau}\|_{S^e} \lesssim \tau \quad \mu(E_\tau) \lesssim \lambda^{p'} \tau^{-p'} |D|. \quad (1.75)$$

Let us select  $K_\lambda$  as in (1.72) so that the first condition of (1.74) is satisfied. Fix a  $D \in \mathbb{D}$  and  $\tau > 0$  and let us carry out the tree selection algorithm described in the Section 1.3.4 for the function  $\mathbb{F}\mathbb{1}_{D \setminus K_\lambda}$ . We may assume that  $\tau \lesssim \lambda$  because we have already shown that (1.75) holds with  $E_\tau = \emptyset$  if  $\tau > C\lambda$  with  $C > 1$  large enough.

After finitely many steps  $L$  suppose that we have selected trees  $T_1 \cdots T_L$ . We may assume that for every selected tree it holds that  $I_{T_j} \subset I_D$  and  $I_{T_j} \not\subset I_n$ . Let us now consider the multi-frequency Calderón-Zygmund decomposition of  $f\mathbb{1}_{I_D}$  as follows. for each  $I_n \subset I_D$  let  $\Xi_n = \{\xi_{T_j} \in \omega_{T_j} : I_{T_j} \supseteq I_n\}$  be the frequencies of the tops of the trees  $T_j$  that intersect  $I_n$ . We write

$$f\mathbb{1}_D = g_c + \sum_n g_n + \sum_n b_n \quad (1.76)$$

where

$$g_c = f\mathbb{1}_{D \setminus \bigcup_n I_n} \quad g_n(x) = \sum_{\xi \in \Xi_n} \langle f\mathbb{1}_D; w_{I_n \times B_{|I_n|^{-1}}(\xi)} \rangle w_{I_n \times B_{|I_n|^{-1}}(\xi)}(x) |I_n| \quad (1.77)$$



and  $b_n = f\mathbb{1}_D - g_n$ . Notice that

$$\begin{aligned} |I_n|^{-1/2}\|g_n\|_{L^2} &\leq |I_n|^{-1/2}\|f\mathbb{1}_{I_n}\|_{L^2} \\ |I_n|^{-1/2}\|g_n\|_{L^2} &\leq |\Xi_n|^{1/2}|I_n|^{-1}\|f\mathbb{1}_{I_n}\|_{L^1} \end{aligned}$$

and by interpolation and maximality of  $I_n$  we obtain

$$|I_n|^{-1/2}\|g_n\|_{L^2} \lesssim |\Xi_n|^{1/2-1/p'}|I_n|^{-1/p'}\|f\mathbb{1}_{I_n}\|_{L^p} \lesssim |\Xi_n|^{1/2-1/p'}\lambda$$

while  $|g_c| < \lambda$  by construction of  $K_\lambda$ .

The bad part  $b = \sum_n b_n$  is orthogonal to the wave packets in the selected trees:

$$P \in \bigcup_{j=1}^L T_j \setminus K_\lambda \quad \implies \quad \langle \sum_n b_n; w_P \rangle = 0.$$

so from the weak  $L^2$  bounds for the selection it follows that

$$\sum_{j=1}^L |I_{T_j}| \lesssim \tau^{-2} \left( \int_{I_D} |g_c(x)|^2 dx + \sum_{n: I_n \subset I_D} \int_{I_n} |g_n(x)|^2 dx \right).$$

Let  $N(x) = \sum_{j=1}^L \mathbb{1}_{I_{T_j}}$  be the counting function of the top intervals of the trees so that  $\sum_{j=1}^L |I_{T_j}| = \int_{I_D} N(x) dx$ . On each  $I_n$  the function  $N$  is constant on each  $I_n$  so we obtain

$$\begin{aligned} \int_{I_D} (1 + N(x)) dx &\lesssim \lambda^2 \tau^{-2} \int_{I_D} (1 + |N(x)|^{1-2/p'}) dx \\ &\lesssim \lambda^2 \tau^{-2} |I_D|^{2/p'} \left( \int_{I_D} 1 + |N(x)| dx \right)^{1-2/p'} \end{aligned}$$

and thus

$$\sum_{j=1}^L |I_{T_j}| \lesssim \int_{I_D} N(x) dx \lesssim \lambda^{p'} \tau^{-p'} |I_D|$$

as required.

For the case  $p \in [2, \infty]$  we need to prove it is sufficient to replicate the above argument and using  $\{x: M_2 f(x) > \lambda\}$  to construct  $K_\lambda$ .

### 1.4.2 Proof of iterated bounds for $\mathbb{A}$

The proof of the bounds (1.68) also relies on a certain projection property. As we have done in Section 1.3.5 we show the following bounds for the auxiliary embedding  $\mathbb{M}$  given by (1.57):

$$\begin{aligned} \|\mathbb{M}\|_{L^\infty \mathcal{E}^\infty S^\infty} &\lesssim \|a\|_{L^\infty} & \|\mathbb{M}\|_{L^\infty \mathcal{E}^1, \infty S^\infty} &\lesssim \|a\|_{L^\infty} \\ \|\mathbb{M}\|_{L^1, \infty \mathcal{E}^\infty S^\infty} &\lesssim \|a\|_{L^1} & \|\mathbb{M}\|_{L^1, \infty \mathcal{E}^1, \infty S^\infty} &\lesssim \|a\|_{L^1} \end{aligned}$$

and then we conclude by interpolation and using Lemma 1.14.

The first bound follows trivially since we have already shown in 1.3.5 that  $\|\mathbb{M}\|_{S^\infty} \lesssim \|a\|_{L^\infty}$ . Similarly We have that  $\mathbb{M}$  is local so that for any  $D$

$$\|\mathbb{M}\mathbb{1}_D\|_{L^1, \infty(S^\infty)} \lesssim \|a\mathbb{1}_D\|_{L^1}.$$

In fact for  $I_P \subset I_D$ ,  $\mathbb{M}(P)$  depends on the values of  $a$  only on  $I_D$ . This implies the bound

$$\|\mathbb{M}\|_{\mathcal{E}^{1,\infty}(S^\infty)} = \sup_{D \in \mathbb{D}} \frac{\|\mathbb{M}\mathbb{1}_D\|_{L^{1,\infty}(S^\infty)}}{|I_D|} \lesssim \|a\|_{L^\infty}.$$

If  $p = 1$  then let us set

$$\begin{aligned} K_\lambda &= \bigcup_{n \in \mathbb{N}} D_n = \bigcup_{n \in \mathbb{N}} D(I_n) \quad \{x: M_1 a(x) > \lambda\} = \bigcup_{n \in \mathbb{N}} I_n \\ &\implies \nu(K_\lambda) \lesssim \sum_{n \in \mathbb{N}} |I_n| \lesssim \lambda^{-p} \|f\|_{L^p}^p \end{aligned}$$

where  $I_n$  are the maximal dyadic intervals with respect to inclusion contained in  $\{x: M_1 a(x) > \lambda\}$ . Consider the ‘‘stopped’’ function

$$\tilde{a}(x) = a(x)\mathbb{1}_{W_2 \setminus \bigcup_n I_n}(x) + \sum_n \mathbb{1}_{I_n}(x) \int_{I_n} a(y) dy$$

that by maximality of  $I_n$  satisfies  $|\tilde{a}(x)| \lesssim \lambda$ . Using definition (1.57) it follows that

$$\mathbb{M}(P) = \tilde{\mathbb{M}}(P) \quad \forall P: I_P \not\subset \bigcup_n I_n$$

where

$$\tilde{\mathbb{M}}(P) = \frac{1}{|I_P|} \int_{I_P} |\tilde{a}(x)| \mathbb{1}_{\omega_P}(c(x)) dx.$$

Thus it follows from the reasoning for  $p = \infty$  that for

$$\|\mathbb{M}\mathbb{1}_{P \setminus K_\lambda}\|_{L^\infty S^\infty} \lesssim \|\tilde{a}\|_{L^\infty} \lesssim \lambda \quad \|\mathbb{M}\mathbb{1}_{P \setminus K_\lambda}\|_{L^{1,\infty} S^\infty} \lesssim \|\tilde{a}\|_{L^\infty} \lesssim \lambda$$

as required.

### 1.4.3 Sparse domination

We now show how the bounds (1.67) and (1.68) can be used to obtain sparse bounds for the Walsh Carleson operator. By this we mean

$$\left| \sum_{P \in \mathfrak{P}} \langle f; w_P \rangle \langle a(\cdot); \mathbb{1}_{\omega_{P_d}}(c(\cdot)) w_P(\cdot) \rangle |I_P| \right| \lesssim \sup_{\mathfrak{S}} \sum_{I \in \mathfrak{S}} |I| \left( \int_I |f|^s \right)^{1/s} \int_I a \quad (1.78)$$

for any  $s > 1$ . The supremum is taken over all sparse grids  $\mathfrak{S}$  i.e. collections of intervals that satisfy bounds (0.10). We have shown in 1.2.3 that the expression on the right is uniformly bounded in  $L^p \times L^{p'}$  for all  $p \in (s, \infty)$ .

We begin by noticing that the bounds (1.67) and (1.68) can be rewritten as follows. Fix  $s > 1$ ,  $q \in (s', \infty)$ , and  $q' \in (1, \infty)$  such that  $\frac{1}{q} + \frac{1}{q'} = 1$ . Given any strip  $D \in \mathbb{D}$  there exists a subset  $K_D \subset D$  such that

$$\begin{aligned} \frac{1}{|I_D|^{1/s}} \|\mathbb{F}\mathbb{1}_{D \setminus K_D}\|_{L_\mu^q S^e} &\lesssim \frac{1}{|I_D|^{1/s}} \|f\mathbb{1}_{I_D}\|_{L^s} \\ \frac{1}{|I_D|} \|\mathbb{A}\mathbb{1}_{D \setminus K_D}\|_{L_\mu^{q'} S^e} &\lesssim \frac{1}{|I_D|} \|a\mathbb{1}_{I_D}\|_{L^1} \end{aligned} \quad (1.79)$$

and with

$$|\nu(K_D)| \leq \varepsilon \nu(D) \quad \text{for some } \varepsilon < 1. \quad (1.80)$$

This follows by applying the iterated embedding bounds to the functions  $f\mathbb{1}_{I_D}$  and  $a\mathbb{1}_{I_D}$  respectively and setting  $K = K_D^{(f)} \cup K_D^{(a)}$  to be union of the exceptional sets for  $f\mathbb{1}_D$  and  $a\mathbb{1}_D$ . While this is not strictly necessary, recall that

$$\begin{aligned} K_D^{(f)} &= \bigcup_n D(I_n) & \bigcup_n I_n &= \left\{ x: M_s f(x) > C|I_D|^{-1/s} \|f\mathbb{1}_{I_D}\|_{L^s} \right\} \\ K_D^{(a)} &= \bigcup_n D(I_n) & \bigcup_n I_n &= \left\{ x: M_1 a(x) > C|I_D|^{-1} \|a\mathbb{1}_{I_D}\|_{L^1} \right\} \end{aligned}$$

for some  $C \gg 1$  be a large enough. By a limiting procedure we can suppose that the sum on the left hand side of (1.78) is taken over a finite collection of tiles. Set  $\mathfrak{S}_0 = I_0$  for some  $I_0$  large enough so that it contains all intervals  $I_P$  of the finite collection of tiles. Iteratively define

$$\mathfrak{S}_{n+1} = \bigcup_{I \in \mathfrak{S}_n} \{J \in \mathfrak{J}_{K_{D(I)}}\}$$

where  $\mathfrak{J}_{K_D}$  is a family of intervals that generates a covering of the exceptional set  $K_D$  i.e.

$$\sum_{J \in \mathfrak{J}_{K_D}} |J| \approx \nu(K_D) \quad \bigcup_{J \in \mathfrak{J}_{K_D}} D(J) \supset K_D$$

and  $J \in \mathfrak{J}_{K_D}$  may be taken pairwise disjoint. Let us then set  $\mathfrak{S} = \bigcup_{n=0}^{\infty} \mathfrak{S}_n$  so that it holds that

$$\sum_{P \in \mathbb{P}} \mathbb{F}(P) \mathbb{A}(P) |I_P| = \sum_{n=0}^{\infty} \sum_{I \in \mathfrak{S}_n} \sum_{P \in D(I) \setminus \bigcup_{J \in \mathfrak{S}_{n+1}} D(J)} \mathbb{F}(P) \mathbb{A}(P) |I_P|.$$

Using (1.79), outer-measure Hölder inequality and the fact that by construction

$$K_{D(I)} \subset \bigcup_{J \in \mathfrak{S}_{n+1}} D(J)$$

it holds that

$$\begin{aligned} \left| \sum_{P \in \mathbb{P}} \mathbb{F}(P) \mathbb{A}(P) |I_P| \right| &\lesssim \sum_{n=0}^{\infty} \sum_{I \in \mathfrak{S}_n} |I| \frac{1}{|I|^{1/s}} \|f\mathbb{1}_I\|_{L^s} \frac{1}{|I|} \|a\mathbb{1}_I\|_{L^1} \\ &= \sum_{I \in \mathfrak{S}} |I| \left( \int_I |f|^s \right)^{1/s} \int_I a \end{aligned}$$

as required.

It remains to check that  $\mathfrak{S}$  is sparse. Suppose that  $I \in \mathfrak{S}_n$ , then

$$\sum_{\substack{J \in \mathfrak{S} \\ J \subset I}} |J| = \sum_{m=n+1}^{\infty} \sum_{\substack{J \in \mathfrak{S}_m \\ J \subset I}} |J| \lesssim \sum_{m=n+1}^{\infty} \varepsilon^{m-n} |I| \lesssim |I|.$$

Since given any two intervals  $J, J' \in \mathfrak{S}$  either they are disjoint or one is contained inside the other the above bounds follows for any dyadic  $I$ .



## Chapter 2

# Variational Carleson embeddings into the upper 3-space

Computations are like parsley:  
they go well with anything.  
—P.A.

This chapter contains the result of the paper [Ura16].

In this chapter we formulate embedding maps into time-frequency space related to the Carleson operator and its variational counterpart. We prove bounds for these embedding maps by iterating the outer measure theory of [DT15]. Introducing iterated outer  $L^p$  spaces is a main novelty of this paper.

### 2.1 Introduction

In this paper we consider the Carleson Operator

$$\mathcal{C}_c f(z) := \int_{c(z)}^{+\infty} \widehat{f}(\xi) e^{i\xi z} d\xi, \quad (2.1)$$

with  $c : \mathbb{R} \rightarrow \mathbb{R}$  a Borel-measurable stopping function. The Variational Carleson Operator studied by Oberlin et al. in [OSTTW12] is given by:

$$\mathcal{V}^r \mathcal{C}_c f(z) = \left( \sum_{k \in \mathbb{Z}} |\mathcal{C}_{c_{k+1}} f(z) - \mathcal{C}_{c_k} f(z)|^r \right)^{1/r} \quad (2.2)$$

where  $\mathbf{c} : \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  is a stopping sequence of Borel-measurable functions such that  $\mathbf{c}_k(z) \leq \mathbf{c}_{k+1}(z)$  for all  $z \in \mathbb{R}$  and  $k \in \mathbb{Z}$ . The boundedness on  $L^p(\mathbb{R})$  with  $p \in (1, \infty)$  of these operators, uniformly with respect to the stopping functions  $c$  and  $\mathbf{c}$ , implies the famous Carleson Theorem on the almost everywhere convergence of the Fourier integral for functions in  $L^p(\mathbb{R})$ . The main technique for bounding these operators were first introduced by Carleson in his paper [Car66] on the convergence of Fourier series for  $L^2([-\pi/2, \pi/2])$  functions and is often referred to as time-frequency analysis.

The purpose of this paper is to discuss embedding maps into time-frequency space  $\mathbb{X} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$  relevant to (2.1) and (2.2). In Theorems 2.1, 2.2, and 2.3 we show the boundedness properties of these embedding maps in terms of appropriately defined norms. Generally speaking an embedding map is a representation of a function by another function defined on the symmetry group of the problem at hand. The appropriate norms for dealing with these embedded functions are the outer measure  $L^p$  norms introduced in [DT15] in the context of the Bilinear Hilbert Transform, an operator with the same symmetries as (2.1) and (2.2).

Theorem 2.2 is an extension of the result of [DT15] to  $1 < p < 2$ . For our proof we introduce iterated, or semi-direct product, outer measure  $L^p$  spaces and incorporate the idea by Di Plinio and Ou [DPO15] of using multi-frequency Calderón-Zygmund theory from [NOT10]. The embedding Theorems 2.1 and 2.3 are somewhat dual to 2.2 for the purpose of bounding the bilinear form associated to (2.1) and (2.2) respectively.

In [OSTTW12] the operator (2.2) has been shown to be bounded for  $p \in (1, \infty)$  and  $r \in (2, p')$ . The proof in the range  $p \in (2, r)$  requires only theorems that make use of non-iterated outer measure spaces of [DT15]. While initially introduced only to address the range  $p \in (r', 2]$ , iterated outer measure spaces surprisingly provide a direct proof in the complete range  $p \in [r, \infty)$ , and hereby explain ad-hoc interpolation techniques used in [OSTTW12].

The advantage of reasoning in terms of embedding maps is also attested by the recent developments in [CDPO16] that prove sharp weighted bounds for the Bilinear Hilbert Transform using the embedding from [DPO15]. In a similar spirit, the embedding maps and the results of the present paper are used to obtain sparse domination and weighted boundedness for the Variational Carleson Operator by Di Plinio, Do, and the author in [DPDU16].

We also point out the recent paper [DMT17] in which Do, Muscalu, and Thiele use outer-measure  $L^p$  spaces to provide variational bounds for bilinear Fourier inversion integrals, that are bilinear versions of (2.2).

On a historical note, we point out Hunt's extension [Hun68] to  $L^p$  with  $p \in (1, \infty)$  of Carleson's pointwise almost-everywhere convergence result [Car66] for Fourier series of functions on  $L^2([-\pi/2, \pi/2])$ . Carleson's and

Hunt's results depend on a fine analysis of the properties of a function on the torus. In [Fef73] Fefferman concentrated on proving the same result by a careful study of the operator (2.1). The wave-packet representation for the operator that is crucial for making use of embedding maps appeared in [LT00] that provides a more symmetric approach encompassing the aforementioned two ideas. This approach inspired both [OSTTW12] and the present paper.

Finally, we emphasize that we formulate an embedding map into the time-frequency space parameterized by continuous parameters, in the vein of [DT15]. This allows us to avoid model-sum operators and averaging procedures ubiquitous in other works in time-frequency analysis. Furthermore, such a formulation proves to be more versatile and in particular the results of the present paper imply all the bounds for the discretized model used in [OSTTW12].

### 2.1.1 The Carleson operator

For simplicity we begin by discussing the Carleson operator (2.1) that is a specific instance of (2.2) for  $r = +\infty$ . The operator is given pointwise by the Fourier multiplier operator associated to the multiplier  $\mathbb{1}_{[c(z), +\infty)}(\xi)$  applied to  $f$ . This can be expressed in terms of a wavelet frame

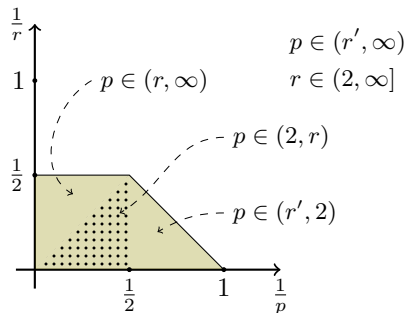


Figure 2.1: Bounds of  $\mathcal{V}^r \mathcal{C}_c$  on  $L^p(\mathbb{R})$ .

centered at frequency  $c(z)$  using a continuous Littlewood-Paley decomposition:

$$\mathcal{C}_c f(z) = \int_{\mathbb{R}^+} \int_{\mathbb{R}} f * \psi_{\eta,t} * \psi_{\eta,t}(z) \chi(t(\eta - c(z))) \, d\eta dt \quad (2.3)$$

where

$$\psi_{\eta,t}(z) := t^{-1} e^{i\eta z} \psi\left(\frac{z}{t}\right) \quad (2.4)$$

with  $\psi \in S(\mathbb{R})$  a suitably normalized, non-negative, even, generating wavelet with Fourier transform  $\widehat{\psi}$  supported in a small ball  $B_b$ . We use the notation  $B_r(x) := (x - r, x + r)$  to denote a ball of radius  $r$  centered at  $x$ , while if  $x = 0$  we omit it by simply writing  $B_r$ . The non-negative cutoff function  $\chi$  satisfies

$$\chi \in C_c^\infty(B_\varepsilon(d)) \quad B_\varepsilon(d) \subset (b, +\infty) \quad \int \chi = 1. \quad (2.5)$$

Given two functions  $f, a \in S(\mathbb{R})$  set

$$F(y, \eta, t) := f * \psi_{\eta,t}(y) \quad (2.6)$$

$$A(y, \eta, t) := \int_{\mathbb{R}} a(z) \psi_{\eta,t}(y - z) \chi(t(\eta - c(z))) \, dz. \quad (2.7)$$

The arguments of the above functions are points of the time-frequency space  $\mathbb{X} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$  that parameterizes the defining symmetries of the class of operators defined by (2.1) i.e. translation of the function, translation of its Fourier transform, and dilation. The outer measure  $L^p$  spaces allow one to deal with the overdermanancy of the wave-packets.

The wave packet representation (2.3) gives the inequality

$$\left| \int_{\mathbb{R}} \mathcal{C}_c f(z) a(z) \, dz \right| \leq \left| \iiint_{\mathbb{X}} F(y, \eta, t) A(y, \eta, t) \, dy d\eta dt \right|. \quad (2.8)$$

By duality the bound of the operator (2.1) on  $L^p(\mathbb{R})$  follows from bounds on  $L^p(\mathbb{R}) \times L^{p'}(\mathbb{R})$  of the bilinear form on the left hand side of the previous display.

The abstract framework of outer measure  $L^p$  spaces provides us with the Hölder type bound

$$\left| \iiint_{\mathbb{X}} F(y, \eta, t) A(y, \eta, t) \, d\eta dy dt \right| \lesssim \|F\|_{L^p \mathbb{E}^q(S_c)} \|A\|_{L^{p'} \mathbb{E}^{q'}(S_m)} \quad (2.9)$$

with  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{q} + \frac{1}{q'} = 1$ . Appearing on the right are iterated outer  $L^p$  quasi-norms that we elaborate on in Section 2.2.

The embedding maps defined via equations (2.7) and (2.6), that we call “mass” and “energy” embeddings for historical reasons (compare with [LT00]), satisfy the bounds

$$\|A\|_{L^{p'} \mathbb{E}^{q'}(S_m)} \lesssim \|a\|_{L^{p'}}, \quad (2.10)$$

$$\|F\|_{L^p \mathbb{E}^q(S_c)} \lesssim \|f\|_{L^p}. \quad (2.11)$$

**Theorem 2.1** (Mass embedding bounds). *For any  $p' \in (1, \infty]$ ,  $q' \in (1, \infty]$ , and for any function  $a \in L^{p'}(\mathbb{R})$  the bounds (2.10) for the embedding (2.7) hold with a constant independent of the Borel measurable function  $c : \mathbb{R} \rightarrow \mathbb{R}$ .*

**Theorem 2.2** (Energy embedding bounds). *For any  $p \in (1, \infty]$ ,  $q \in (\max(2; p'), \infty]$ , and for any  $f \in L^p(\mathbb{R})$  the bounds (2.11) for the embedding (2.6) hold.*

Theorem 2.1 follows as a corollary of Theorem 2.3 below while Theorem 2.2 will be proven in Section 2.6.

The boundedness of the Carleson Operator on  $L^p(\mathbb{R})$  follow as a result of the above discussion. Indeed for any  $p, p' \in (1, \infty)$  with  $\frac{1}{p} + \frac{1}{p'} = 1$  one can find  $q, q' \in (1, \infty)$  such that  $\frac{1}{q} + \frac{1}{q'} = 1$  and bounds (2.10) and (2.11) hold.

We remark that iterated outer measure spaces are used to address the case  $p \in (1, 2)$ . In Section 2.6 we show that if  $p \in (2, \infty)$  a the non-iterated version of outer measure  $L^p$  spaces are sufficient to prove  $L^p$  boundedness of (2.1).

### 2.1.2 The variational Carleson operator

The operator (2.2), introduced and studied in [OSTTW12], is bounded on  $L^p(\mathbb{R})$  for  $r \in (2, \infty)$  and  $p \in (r', \infty)$ . The above paper also shows that this range is sharp in the sense that that strong  $L^p$  bounds do not hold outside this range (see Figure 2.1).

By duality it is sufficient to prove the bilinear a priori bound

$$\left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbf{a}_k(z) \int_{\mathbf{c}_k(z)}^{\mathbf{c}_{k+1}(z)} \widehat{f}(\xi) e^{i\xi z} d\xi dz \right| \lesssim \|f\|_{L^p} \|\mathbf{a}\|_{L^{p'}(l^{r'})}. \quad (2.12)$$

with a constant independent of the stopping sequence  $\mathbf{c}$ . For the above expression to make sense we require that  $f \in S(\mathbb{R})$  while  $\mathbf{a} \in L^{p'}(l^{r'})$  i.e.  $z \mapsto \mathbf{a}(z) = (\mathbf{a}_k(z))_{k \in \mathbb{Z}}$  is a function on  $\mathbb{R}$  such that for every  $z \in \mathbb{R}$  its value is the sequences  $\mathbf{a}(z) = (\mathbf{a}_k(z))_{k \in \mathbb{Z}} \in l^{r'}(\mathbb{Z})$ . The function  $\mathbf{a}$  is Borel measurable in Bochner sense and

$$\|\mathbf{a}\|_{L^{p'}(l^{r'})} := \left( \int_{\mathbb{R}} \|\mathbf{a}(z)\|_{l^{r'}}^{p'} dz \right)^{1/p'} < \infty.$$

Analogously to (2.8), the left had side of (2.12) admits a wave-packet domination

$$\left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbf{a}_k(z) \int_{\mathbf{c}_k(z)}^{\mathbf{c}_{k+1}(z)} \widehat{f}(\xi) e^{i\xi z} d\xi dz \right| \leq \iiint_{\mathcal{X}} |F(y, \eta, t) \mathbb{A}(y, \eta, t)| dy d\eta dt. \quad (2.13)$$

where the embedding map  $\mathbf{a} \mapsto \mathbb{A}$  is given by

$$\mathbb{A}(y, \eta, t) := \sup_{\Psi} \left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbf{a}_k(z) \Psi_{y, \eta, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dz \right|. \quad (2.14)$$

The supremum above is taken over all possible choices of left or right truncated wave packets  $\Psi_{y, \eta, t}^{c_-, c_+}$ . A left truncated wave packet  $\Psi_{y, \eta, t}^{c_-, c_+}$  at  $(y, \eta, t) \in \mathcal{X}$  is a  $S(\mathbb{R})$  function parameterized by  $c_- < c_+ \in \mathbb{R} \cup \{+\infty\}$ . The parameterization satisfies the properties below. The following three functions of the variable  $z$

$$\begin{aligned} & e^{-i\eta(y+tz)} t \Psi_{y, \eta, t}^{c_-, c_+}(y+tz) \\ & t^{-1} \partial_{c_-} \left( e^{-i\eta(y+tz)} \Psi_{y, \eta, t}^{c_-, c_+}(y+tz) \right) \\ & t^{-1} \partial_{c_+} \left( e^{-i\eta(y+tz)} t \Psi_{y, \eta, t}^{c_-, c_+}(y+tz) \right) \end{aligned} \quad (2.15)$$

are bounded in  $S(\mathbb{R})$  uniformly for all  $(y, \eta, t) \in \mathcal{X}$  and  $c_- < c_+ \in \mathbb{R}$ . For some constant  $b > 0$  the functions  $\Psi_{y, \eta, t}^{c_-, c_+}$  satisfy

$$\text{spt } \widehat{\Psi}_{y, \eta, t}^{c_-, c_+} \subset B_{t^{-1}b}(\eta). \quad (2.16)$$



For some constants  $d, d', d'' > 0$ , and  $\varepsilon > 0$  it holds that

$$\Psi_{y,\eta,t}^{c_-,c_+} \neq 0 \quad \text{only if} \quad \begin{cases} t(\eta - c_-) \in B_\varepsilon(d) \\ t(c_+ - \eta) > d' > 0 \end{cases} \quad (2.17)$$

$$\Psi_{y,\eta,t}^{c_-,c_+} = \Psi_{y,\eta,t}^{c_-,+\infty} \quad \text{if} \quad t(c_+ - \eta) > d'' > d' > 0. \quad (2.18)$$

The wave packet  $\Psi_{y,\eta,t}^{c_-,c_+}$  is right truncated if  $\Psi_{y,-\eta,t}^{-c_+,-c_-}$  is left truncated.

The main result of this paper is the following bounds for the embedding (2.14) that are analogous to the bounds (2.10).

**Theorem 2.3** (Variational mass embedding bounds). *For any  $r' \in [1, 2)$ ,  $p' \in (1, \infty]$ , and  $q' \in (r', \infty]$  and any function  $\mathbf{a} \in L^{p'}(I^{r'})$  the function  $\mathbb{A}$  defined by (2.14) satisfies the bounds*

$$\|\mathbb{A}\|_{L^{p'}E^{q'}(S_m)} \lesssim \|\mathbf{a}\|_{L^{p'}(I^{r'})} \quad p' \in (1, \infty] \quad q' \in (r', \infty]; \quad (2.19)$$

furthermore the weak endpoint bounds

$$\begin{aligned} \|\mathbb{A}\|_{L^{p'}E^\infty(S_m)} &\lesssim \|\mathbf{a}\|_{L^{p'}(I^{r'})} & p' \in (1, \infty] \\ \|\mathbb{A}\|_{L^{1,\infty}E^{q'}(S_m)} &\lesssim \|\mathbf{a}\|_{L^1(I^{r'})} & q' \in (r', \infty] \\ \|\mathbb{A}\|_{L^{1,\infty}E^{r',\infty}(S_m)} &\lesssim \|\mathbf{a}\|_{L^1(I^{r'})} \end{aligned} \quad (2.20)$$

hold. All the above inequalities hold with constants independent of the stopping sequence  $\mathbf{c}$  appearing in (2.14).

We refer to Section 2.2 for the description of the outer measure structure on  $\mathbb{X}$  and for the precise definition of the iterated outer measure  $L^p$  norms appearing on the left hand sides.

**Corollary 2.4** (Boundedness of the variational Carleson operator [OSTTW12]). *The operator (2.2) defined pointwise for  $f \in S(\mathbb{R})$  extends to a bounded operator on  $L^p(\mathbb{R})$  for  $r \in (2, \infty]$  and  $p \in (r', \infty)$ .*

Given Theorem 2.3 the above can be obtained analogously as for the operator (2.1). For for  $p$  and  $r$  set  $\frac{1}{p'} = 1 - \frac{1}{p}$ ,  $\frac{1}{r'} = 1 - \frac{1}{r}$ , and choose  $q$  and  $q'$  so that  $\frac{1}{q} + \frac{1}{q'} = 1$  and the bounds (2.11) and (2.19) hold. Using the outer measure Hölder inequality (2.9) with the variational embedded function  $\mathbb{A}$  in lieu of  $A$  and the wave-packet representation (2.13) we obtain the required bound (2.12).

Theorem 2.1 follows from from Theorem 2.3 when  $r = \infty$  by formally setting

$$\mathbf{a}_k(z) = \begin{cases} a(z) & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \quad \mathbf{c}_k(z) = \begin{cases} -\infty & \text{if } k < 0 \\ c(z) & \text{if } k = 0 \\ +\infty & \text{if } k > 0 \end{cases} \quad (2.21)$$

In particular the term  $\psi_{\eta,t}(y-z)\chi(t(\eta - c_-))$  appearing in (2.7) are left truncated wave packets with respect to the parameters  $c_-$  and  $c_+ = +\infty$ .

### 2.1.3 Structure of the paper

The rest of this paper is organized as follows. In Section 2.2 we define the outer measure structure on  $\mathbb{X}$ . We then recall properties of outer measure  $L^p$  spaces and generalize them to the iterated

construction. In addition we illustrate a limiting argument for maps to outer measure  $L^p$  spaces that allows to consider the bounds (2.10), (2.11), and (2.19) as a-priori estimates. We also prove interpolation inequalities that allow us to restrict the proof only to the weak endpoints of the above bounds. Finally, we formulate the abstract outer Hölder inequality and an outer Radon-Nikodym Lemma that imply inequality (2.9).

In Section 2.3 we prove the wave-packet domination bound (2.13). In particular it is shown that one can choose both the geometric parameters of the outer measure space (see Section 2.2) and the parameters of the truncated wave-packets in a compatible way i.e. so that both Theorems 2.2 and 2.3 as well as the conditions (2.16), (2.17) hold. This is done by providing a wave-packet representation for multipliers of the form  $\mathbb{1}_{[c_-, c_+]}$  with  $c_- < c_+ \in \mathbb{R} \cup \{+\infty\}$ . For any stopping sequence  $\mathfrak{c}$  this yields an embedded function  $\mathbb{A}_{\mathfrak{c}}(y, \eta, t)$  so that

$$\int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathfrak{a}_k(z) \int_{\mathfrak{c}_k(z)}^{\mathfrak{c}_{k+1}(z)} \widehat{f}(\xi) e^{i\xi z} d\xi dz = \iiint_{\mathfrak{X}} F(y, \eta, t) \mathbb{A}_{\mathfrak{c}}(y, \eta, t) dy d\eta dt. \quad (2.22)$$

The embedded function  $\mathbb{A}_{\mathfrak{c}}$  is pointwise dominated by  $\mathbb{A}$  and the map  $\mathfrak{a} \mapsto \mathbb{A}_{\mathfrak{c}}$  is shown to be linear. Furthermore the same procedure shows that the inequality in (2.8) is actually an equality i.e.

$$\int_{\mathbb{R}} C_c f(z) a(z) dz = \iiint_{\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+} F(y, \eta, t) A(y, \eta, t) dy d\eta dt. \quad (2.23)$$

In Section 2.4 we introduce an auxiliary embedding map for which we show iterated outer measure bounds. The crucial result is given by the covering Lemma 2.19 that allows one to control the measure of super-level sets of this embedding map and by a projection Lemma 2.22 that implies iterated bounds.

In Section 2.5 we actually prove Theorem 2.3 by showing the the auxiliary embedding map of Section 2.4 dominates the embedding (2.14) in terms of sizes.

Finally, in Section 2.6 we show that bound (2.11) holds: this follows from an adaptation of the results of [DPO15]. We also remark how in the case  $p \in (2, r)$  a non-iterated version of outer measure  $L^p$  spaces is enough to obtain  $L^p$  bounds for (2.2) and thus for (2.1) with  $p \in (2, \infty)$ .

## 2.1.4 Notation

We quickly recall some useful notation.

We say that  $A(x) \lesssim B(x)$  if there exists a constant  $C > 0$  such that  $A(x) \leq CB(x)$  for all  $x$  in the domains of  $A$  and  $B$  respectively. Unless otherwise specified the constant  $C > 0$  is absolute. We may emphasize the dependence on a specific parameter  $p$  by writing  $A(x) \lesssim_p B(x)$ . We write  $A(x) \approx B(x)$  if  $A(x) \lesssim B(x)$  and  $A(x) \gtrsim B(x)$ .

We denote open and close Euclidean balls of  $\mathbb{R}$  as

$$B_r(x) := (x - r, x + r) \quad B_r := (-r, +r) \quad \overline{B_r(x)} := [x - r, x + r] \quad \overline{B_r} := [-r, +r].$$

We indicate by  $\mathbb{1}_{\Theta}$  the characteristic function of the set  $\Theta$  i.e.

$$\mathbb{1}_{\Theta}(x) := \begin{cases} 1 & \text{if } x \in \Theta \\ 0 & \text{if } x \notin \Theta \end{cases}$$

For an arbitrary large  $N > 0$  we introduce the smooth bump function

$$W(z) := (1 + |z|^2)^{-N/2} \quad W_t(z) := t^{-1} W\left(\frac{z}{t}\right). \quad (2.24)$$

We define

$$\oint_{B_r(x)} f(z) dz := \frac{1}{2r} \int_{B_r(x)} f(z) dz.$$

The operators  $M$  and  $M_p$  are the Hardy-Littlewood maximal function i.e.

$$\begin{aligned} Mf(z) &:= \sup_{t \in \mathbb{R}^+} \oint_{B_t(z)} |f(z')| dz' \\ M_p f(z) &:= \sup_{t \in \mathbb{R}^+} \left( \oint_{B_t(z)} |f(z')|^p dz' \right)^{1/p}. \end{aligned} \quad (2.25)$$

Given a function  $\varphi \in S(\mathbb{R})$  we obtain its frequency translates and dilates by setting

$$\varphi_{\eta,t}(z) := t^{-1} e^{i\eta z} \varphi\left(\frac{z}{t}\right).$$

The stopping sequence  $\mathbf{c}$  will denote a Borel measurable function defined on  $\mathbb{R}$  with values in increasing sequences in  $\mathbb{R} \cup +\infty$  i.e.

$$z \mapsto \mathbf{c}(z) = (\mathbf{c}_k(z))_{k \in \mathbb{Z}} \quad -\infty < \cdots \leq \mathbf{c}_{k-1}(z) \leq \mathbf{c}_k(z) \leq \mathbf{c}_{k+1}(z) \leq \cdots \leq +\infty.$$

Similarly  $\mathbf{a}$  will denote a Borel Bochner-measurable function on  $\mathbb{R}$  with values in  $l^{r'}$  i.e.

$$z \mapsto \mathbf{a}(z) = (\mathbf{a}_k(z))_{k \in \mathbb{Z}} \in l^{r'}.$$

We use the notation  $L^p(S)$  and  $L^p L^q(S)$  to denote (iterated) outer measure  $L^p$  spaces. The (outer-) measure of the space is omitted from the notation. We distinguish the above from  $L^p$  that are classical Lebesgue spaces. In the case of  $L^p$  spaces on  $\mathbb{R}$  the measure is the Lebesgue measure; when necessary we may emphasize the measure  $\mathcal{L}$  on the space by writing  $L^p_{(\text{d}\mathcal{L})}$ .

## 2.2 Outer measures on the time-frequency space

We begin the description of the outer measure on the time-frequency space  $\mathbb{X}$  by introducing a family of distinguished generating sets. The *tent*  $T(x, \xi, s) \subset \mathbb{X}$  indexed by the top point  $(x, \xi, s) \in \mathbb{X}$  is the set

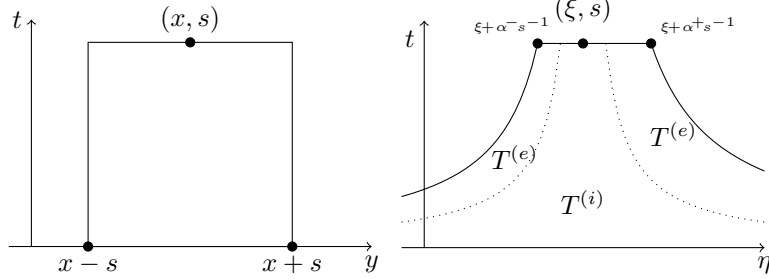
$$\begin{aligned} T(x, \xi, s) &:= T^{(i)}(x, \xi, s) \cup T^{(e)}(x, \xi, s) \\ T^{(i)}(x, \xi, s) &:= \left\{ (y, \eta, t) : |y - x| < s, t(\eta - \xi) \in \Theta^{(i)}, t < s \right\} \\ T^{(e)}(x, \xi, s) &:= \left\{ (y, \eta, t) : |y - x| < s, t(\eta - \xi) \in \Theta^{(e)}, t < s \right\} \end{aligned} \quad (2.26)$$

where

$$\Theta = (\alpha^-, \alpha^+) \quad \Theta^{(i)} = (\beta^-, \beta^+) \quad \Theta^{(e)} = \Theta \setminus \Theta^{(i)} \quad (2.27)$$

are geometric intervals such that  $0 \in \Theta^{(i)} \subseteq \Theta$  i.e.  $\alpha^- \leq \beta^- < 0 < \alpha^+ \leq \beta^+$ . We refer to  $T^{(i)}$  and  $T^{(e)}$  as the interior and exterior parts of the tent  $T$ . To define the iterated outer measure structure we introduce *strips*  $D(x, s) \subset \mathbb{X}$  as

$$D(x, s) := \{(y, \eta, t) : |y - x| < s, t < s\}. \quad (2.28)$$

Figure 2.2: The tent  $T(x, \xi, s)$ .

We indicate the family of all tents by  $\mathbb{T}$  and the family of all strips by  $\mathbb{D}$ . The specific values of the geometric intervals  $\Theta$ ,  $\Theta^{(i)}$ , and  $\Theta^{(e)}$  in (2.26) are often inessential. However, the freedom of choosing appropriate parameters was shown to be important in [DT15]. Theorem 2.3 holds as long as

$$\overline{B_b} \subset \Theta^{(i)} \subset \overline{B_{d'}} \quad \overline{B_\varepsilon(d)} \cup \overline{B_\varepsilon(-d)} \subset \Theta^{(e)} \quad (2.29)$$

with  $b$ ,  $d$ ,  $d'$ , and  $\varepsilon$  appearing in (2.16), (2.17), and (2.18). As a matter of fact, if one were to consider only left truncated wave-packets in (2.14) then Theorem 2.3 would hold as long as

$$B_b \subset \Theta^{(i)} \quad -d' < \beta^- \quad \overline{B_\varepsilon(d)} \subset \Theta^{(e)} \cap \mathbb{R}^+ = [\beta^+, \alpha^+]. \quad (2.30)$$

Theorem 2.1 holds as long as satisfies

$$B_b \subset \Theta^{(i)} \quad \text{spt } \chi \subset \Theta^{(e)} \cap \mathbb{R}^+ = [\beta^+, \alpha^+]. \quad (2.31)$$

Theorem 2.2 holds as long as  $B_b \subset \Theta^{(i)}$ . From now on we will allow all our implicit constants to depend on  $\Theta$  and  $\Theta^{(i)}$ .

We now define the outer measures  $\mu$  and  $\nu$  by introducing the pre-measures  $\bar{\mu}$ , and  $\bar{\nu}$  on the generating sets

$$\bar{\mu}(T(x, \xi, s)) := s \quad \bar{\nu}(D(x, s)) := s. \quad (2.32)$$

The outer measure of an arbitrary subset  $E \subset \mathbb{X}$  are obtained via a covering procedure using *countable* unions of generating sets i.e.

$$\nu(E) := \inf \left\{ \sum_{n \in \mathbb{N}} \bar{\nu}(D_n) : E \subset \bigcup_{\substack{D_n \in \mathbb{D} \\ n \in \mathbb{N}}} D_n, \right\} \quad (2.33)$$

and similarly for  $\mu$  using  $\bar{\mu}$  and the family  $\mathbb{T}$ . We say that  $\nu$  and  $\mu$  are generated by the pre-measures  $(\bar{\nu}, \mathbb{D})$  and  $(\bar{\mu}, \mathbb{T})$  respectively. We call an *outer measure space* a pair  $(\mathbb{X}, \mu)$  of a separable complete measure space  $\mathbb{X}$  and an outer measure  $\mu : 2^\mathbb{X} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ . We will henceforth suppose that  $\mu$  is generated by pre-measures  $(\bar{\mu}, \mathbb{T})$  where  $\mathbb{T}$  is a collection of subsets  $T \subset \mathbb{X}$  that we assume to be Borel measurable.

The final ingredient we need for introducing outer measure  $L^p$  spaces is a notion of how large a function on  $\mathbb{X}$  is. We call a *size* any quasi-norm  $\|\cdot\|_S$  on Borel functions on  $\mathbb{X}$  i.e. a positive functional that satisfies the following properties. **Monotonicity:** for any Borel function  $G_1$  and  $G_2$

$$|G_1| \leq |G_2| \implies \|G_1\|_S \lesssim \|G_2\|_S. \quad (2.34)$$

**Positive homogeneity:** for all Borel functions  $G$

$$\|\lambda G\|_S = |\lambda| \|G\|_S \quad \forall \lambda \in \mathbb{C}. \quad (2.35)$$

**Quasi-triangle inequality:** for any sequence of Borel functions  $G_k$  and for some quasi-triangle constant  $c_s \geq 1$

$$\left\| \sum_{k=0}^{\infty} G_k \right\|_S \leq \sum_{k=0}^{\infty} c_s^{k+1} \|G_k\|_S \quad (2.36)$$

We define the  $S, \mu$  - super-level outer measure as

$$\mu(\|G\|_S > \lambda) := \inf \{ \mu(E_\lambda) : \|G \mathbf{1}_{\mathbb{X} \setminus E_\lambda}\|_S \leq \lambda \} \quad (2.37)$$

where the lower bound is taken over Borel subset  $E_\lambda$  of  $\mathbb{X}$ . The outer- $L^p$  quasi-norms for  $p \in (0, \infty]$  are give by

$$\|G\|_{L^p(S)}^p := \int_{\lambda \in \mathbb{R}^+} p \lambda^p \mu(\|G\|_S > \lambda) \frac{d\lambda}{\lambda};$$

weak outer  $L^p$  quasi-norms are similarly given by

$$\|G\|_{L^{p,\infty}(S)}^p := \sup_{\lambda \in \mathbb{R}^+} p \lambda^p \mu(\|G\|_S > \lambda).$$

The outer  $L^p$  spaces are subspaces of Borel functions on  $\mathbb{X}$  for which the above norms are finite. The expressions defining outer  $L^p$  quasi-norms are based on the super-level set representation of the Lebesgue integral, however the expression  $\mu(\|G\|_S > \lambda)$  that appears in lieu of the classical  $\mu(\{x : |g(x)| > \lambda\})$  cannot always be interpreted as a measure of a specific set. Generally speaking,  $L^p$  spaces for  $p \in (0, \infty)$  are interpolation spaces between the size quasi-norm and the outer measure of the support of a function.

Using a slight abuse of notation we say that a size  $\|\cdot\|_S$  is generated by  $(\|\cdot\|_{S(T)}, \mathbb{T})$  where  $\|\cdot\|_{S(T)}$  are sizes indexed by generating sets  $T \in \mathbb{T}$  and in particular

$$\|G\|_S := \sup_{T \in \mathbb{T}} \|G\|_{S(T)}. \quad (2.38)$$

The construction of iterated outer  $L^p$  spaces is based on using localized versions of outer  $L^q$  quasi-norms as sizes themselves. Notice that outer  $L^q$  norms are quasi-norms since they too satisfy the quasi-triangle inequality. Given a size  $S$  and a generating pre-measure  $(\bar{\nu}, \mathbb{D})$ , outer  $\mathcal{L}^q(S)$  sizes are generated by  $(\mathcal{L}^q(S)(D), \mathbb{D})$  where

$$\|G\|_{\mathcal{L}^q(S)(D)} := \frac{\|F \mathbf{1}_D\|_{L^q(S)}}{\bar{\nu}(D)^{1/q}} \quad (2.39)$$

so  $\|G\|_{\mathcal{L}^q(S)} := \sup_{D \in \mathbb{D}} \|G\|_{\mathcal{L}^q(S)(D)}$ . Consequently we construct iterated outer  $L^p$  spaces as

$$\|G\|_{L^p \mathcal{L}^q(S)}^p := \int_{\tau \in \mathbb{R}^+} p \tau^p \bar{\nu}(\|G\|_{\mathcal{L}^q(S)} > \tau) \frac{d\tau}{\tau}. \quad (2.40)$$

To deal with embedded functions  $F$  and  $\mathbb{A}$  from (2.6) and (2.14) we introduce the respective sizes  $\|\cdot\|_{S_e}$  and  $\|\cdot\|_{S_m}$  that are generated by  $(S_e(T), \mathbb{T})$  and  $(S_m(T), \mathbb{T})$  respectively. The two families of “local” sizes  $S_e(T)$  and  $S_m(T)$  are given by

$$\begin{aligned} \|F\|_{S_e(T)} &:= \frac{\|F \mathbf{1}_{T^{(\epsilon)}}\|_{L^2}}{\mu(T)^{1/2}} + \|F \mathbf{1}_T\|_{L^\infty} \\ &= \|F\|_{S^2(T^{(\epsilon)})} + \|F\|_{S^\infty(T)} \end{aligned} \quad (2.41)$$

$$\begin{aligned} \|\mathbb{A}\|_{S_m(T)} &:= \frac{\|\mathbb{A}\mathbb{1}_T\|_{L^2}}{\mu(T)^{1/2}} + \frac{\|\mathbb{A}\mathbb{1}_{T^{(i)}}\|_{L^1}}{\mu(T)} \\ &= \|\mathbb{A}\|_{S^2(T)} + \|\mathbb{A}\|_{S^1(T^{(i)})}. \end{aligned} \quad (2.42)$$

Here  $L^2$ ,  $L^\infty$ , and  $L^1$  refer to classical Lebesgue  $L^p$  norms on  $\mathbb{X}$  with respect to the Borel measure  $dyd\eta dt$ . The local sizes  $\|\cdot\|_{S_e(T)}$  coincide with the ones introduced on the upper 3-space in [DT15] while  $\|\cdot\|_{S_m(T)}$  are dual to the former in an appropriate sense.

We conclude the construction of outer measure  $L^p$  spaces with a useful remark about the specific geometric properties of coverings with tents  $\mathbb{T}$ . For any tent  $T(x, \xi, s)$  we define its  $R$ -enlargement with  $R > 1$  as

$$RT(x, \xi, s) := \bigcup_{|\xi' - \xi| < Rs^{-1}} T(x, \xi', Rs). \quad (2.43)$$

Notice that  $\mu(RT) \lesssim R^3\mu(T)$  with a constant that depends on the geometric intervals (2.27) but not on  $R$ . As a matter of fact the set  $RT$  can be covered by a finite collection of tents  $T(x, \xi_i, Rs)$  by choosing  $\xi_i$  such that

$$\bigcup_i \{\eta: Rs(\eta - \xi_i) \in \Theta\} \supset \bigcup_{|\xi' - \xi| < Rs^{-1}} \{\eta: Rs(\eta - \xi') \in \Theta\}$$

The number of points  $\xi_i$  needed to do this is bounded up to a constant factor by  $R^2$  and thus  $\mu(RT) \lesssim R^3\mu(T)$ .

### 2.2.1 Properties of outer measure $L^p$ spaces.

We recall some important properties of outer measure  $L^p$  spaces and elaborate on how they carry over to iterated outer-measure spaces. Generally  $\mathbb{X}$  may be any locally compact complete metric space; in our case  $\mathbb{X} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$  with

$$\text{dist}((y, \eta, t); (y', \eta', t')) = t^{-1}|y - y'| + t|\eta - \eta'| + \left|\log \frac{t}{t'}\right|.$$

#### Dominated convergence

While outer measure  $L^p$  spaces fall into the class of quasi-Banach spaces, we record only some functional properties that are useful for our applications.

Recall that the quasi-triangle inequality for sizes (2.36) holds for both finite and infinite sums. Given an outer measure space  $(\mathbb{X}, \mu)$  and a size  $\|\cdot\|_S$ , the outer measure  $L^p$  quasi-norms also satisfy the quasi-triangle inequality:

$$\left\| \sum_{k=0}^{\infty} G_k \right\|_{L^p(S)} \lesssim_{c'_s, c_s, p} \sum_{k=0}^{\infty} c_s'^{k+1} \|G_k\|_{L^p(S)} \quad (2.44)$$

for any  $c'_s > c_s$  where  $c_s$  is the quasi-triangle constant of the size  $S$ . As a matter of fact, for any  $\lambda > 0$  and for every  $k$  choose  $E_{\lambda, k}$  such that

$$\|G_k \mathbb{1}_{\mathbb{X} \setminus E_{\lambda, k}}\|_S \leq \lambda \quad \|G_k\|_{L^p(S)}^p \lesssim \int_{\mathbb{R}^+} p\lambda^p \mu(E_{\lambda, k}) \frac{d\lambda}{\lambda}$$

and set  $E_\lambda = \bigcup_{k=0}^{\infty} E_{\lambda c_s'^{-k-1}, k}$  so that using the quasi-triangle inequality for  $\|\cdot\|_S$  one has

$$\mu(E_\lambda) \leq \sum_{k=0}^{\infty} \mu\left(E_{\lambda c_s'^{-k-1}, k}\right) \quad \left\| \sum_{k=0}^{\infty} G_k \mathbb{1}_{E_\lambda^c} \right\|_S \leq \lambda \frac{c_s}{(c_s' - c_s)}.$$

Thus

$$\begin{aligned} \left\| \sum_{k=0}^{\infty} G_k \right\|_{L^p(S)}^p &\leq p \left( \frac{c_s}{c_s' - c_s} \right)^p \int_{\mathbb{R}^+} \lambda^p \mu(E_\lambda) \frac{d\lambda}{\lambda} \\ &\leq \left( \frac{c_s}{c_s' - c_s} \right)^p \sum_{k=0}^{\infty} c_s'^{p(k+1)} \|G_k\|_{L^p(S)}^p. \end{aligned}$$

If  $p \geq 1$  then this concludes the proof. Otherwise for any  $\varepsilon > 0$  one has

$$\left\| \sum_{k=0}^{\infty} G_k \right\|_{L^p(S)} \lesssim_{\varepsilon, p, c, c_s} \sum_{k=0}^{\infty} (1 + \varepsilon)^{k+1} c_s'^{k+1} \|G_k\|_{L^p(S)}$$

but since  $c_s' > c_s$  was arbitrary this also allows us to conclude.

This fact is crucial to be able to use localized outer  $L^p$  quasi-norms as sizes themselves. Furthermore we deduce the following domination property.

**Corollary 2.5.** *Suppose that  $G$  is a Borel function on  $\mathbb{X}$  and  $|G| \leq \limsup_{n \rightarrow \infty} |G_n|$  pointwise on  $\mathbb{X}$  for some sequence of Borel functions  $G_n$  that satisfy*

$$\|G_{n+1} - G_n\|_{L^p(S)} \leq C c_s'^{-n} \|G_0\|_{L^p(S)} \quad \text{for some } c_s' > c_s.$$

Then

$$\|G\|_{L^p(S)} \lesssim_{C, p, c_s', c_s} \|G_0\|_{L^p(S)}.$$

This follows from (2.44) and from the monotonicity properties of sizes and thus of outer  $L^p$  quasi-norms.

Using this property we will restrict ourselves to proving bounds (2.10), (2.11), and (2.19) for a dense class of functions. In particular we will always consider the functions in play to be smooth and rapidly decaying. For example, given a function  $\mathbf{a} \in L^p(l^{r'})$  one may always choose a sequence of approximating functions  $\mathbf{a}^{(n)} \in C_c^\infty(l^{r'})$  such that

$$\begin{aligned} \|\mathbf{a}^{(0)}\|_{L^p(l^{r'})} &\lesssim \|\mathbf{a}\|_{L^p(l^{r'})} \\ \|\mathbf{a}^{(n+1)} - \mathbf{a}^{(n)}\|_{L^p(l^{r'})} &\lesssim 2^{-Nn} \|\mathbf{a}\|_{L^p(l^{r'})} \end{aligned}$$

for an arbitrary  $N > 1$ . Considering the sequence embedded functions  $\mathbb{A}_n$  associated to  $\mathbf{a}^{(n)}$  via (2.14), the pointwise relation  $\mathbb{A} = \lim_n \mathbb{A}_n$  clearly holds. Corollary 2.5 applied to  $\mathbb{A}_n$  allows us to conclude that if bounds of Theorem 2.3 hold for the functions  $\mathbf{a}^{(n)}$  they also hold for  $\mathbf{a}$ . Thus we can restrict to proving the bounds a priori estimates i.e. we can restrict to showing that they hold for a dense class of functions  $\mathbf{a}$ . The same can be done for the energy embedding bounds of Theorem 2.2.

### Hölder and Radon-Nikodym inequalities

We now illustrate the abstract outer measure results from which inequality (2.9) follows. The first two statements relate to general outer measure spaces and are similar to what was obtained in [DT15].

**Lemma 2.6** (Radon-Nikodym domination). *Consider  $(\mathbb{X}, \mu)$  an outer measure space with  $\mu$ , generated by  $(\bar{\mu}, \mathbb{T})$  as in (2.33), endowed with a size  $\|\cdot\|_S$  generated by  $(\|\cdot\|_{S(T)}, \mathbb{T})$ . Suppose that the generating family  $\mathbb{T}$  consists of Borel sets and satisfies the covering condition i.e.  $\mathbb{X} = \bigcup_{i \in \mathbb{N}} T_i$  for some countable sub-collection  $T_i \in \mathbb{T}$ .*

*If  $\mathcal{L}$  is a positive Borel measure on  $\mathbb{X}$  such that*

$$\int_T |G(P)| d\mathcal{L}(P) \leq C \|G\|_{S(T)} \bar{\mu}(T) \quad \forall T \in \mathbb{T} \quad (2.45)$$

*and for any Borel function  $G$  and*

$$\mu(E) = 0 \implies \mathcal{L}(E) = 0 \quad \forall E \subset \mathbb{X} \text{ Borel} \quad (2.46)$$

*then for any Borel function  $G$  the bound*

$$\left| \int_{\mathbb{X}} G(P) d\mathcal{L}(P) \right| \lesssim \|G\|_{L^1(S)} \quad (2.47)$$

*holds.*

The proof of this Lemma is similar to the one in [DT15].

*Proof.* Suppose  $\|G\|_{L^1(S)} < \infty$ , otherwise there is nothing to prove. For each  $k \in \mathbb{Z}$  let  $E'_{2^k}$  be a Borel set such that

$$\|G \mathbb{1}_{\mathbb{X} \setminus E'_{2^k}}\| \leq 2^k \quad \mu(E'_{2^k}) \leq 2\mu(\|G\|_S > 2^k).$$

so  $\|G\|_{L^1(S)} \lesssim \sum_{k=-\infty}^{+\infty} 2^k \mu(E'_{2^k})$ . Set

$$E_{2^k} := \bigcup_{l=k}^{+\infty} E'_{2^l} \quad \Delta E_{2^k} := E_{2^{k-1}} \setminus E_{2^k} \quad E_0 = \bigcup_{k=-\infty}^{+\infty} E_{2^k} \quad E_\infty = \bigcap_{k=-\infty}^{+\infty} E_{2^k}.$$

We have

$$\left| \int_{\mathbb{X}} G(P) d\mathcal{L}(P) \right| \leq \int_{\mathbb{X} \setminus E_0} |G(P)| d\mathcal{L}(P) + \sum_{k=-\infty}^{+\infty} \int_{\Delta E_{2^k}} |G(P)| d\mathcal{L}(P) + \int_{E_\infty} |G(P)| d\mathcal{L}(P).$$

where

$$\|G \mathbb{1}_{\Delta E_{2^k}}\|_S \leq 2^k \quad \|G\|_{L^1(S)} \lesssim \sum_{k=-\infty}^{+\infty} 2^k \mu(\Delta E_{2^k}).$$

For every  $k$  there exists a countable covering  $\bigcup_{l \in \mathbb{N}} T_{k,l} \supset \Delta E_{2^k}$  such that

$$\sum_{l \in \mathbb{N}} \bar{\mu}(T_{k,l}) \leq 2\mu(\Delta E_{2^k}).$$

For each  $k \in \mathbb{Z}$  apply (2.45) to obtain

$$\int_{\Delta E_{2^k}} |G(P)| d\mathcal{L}(P) \leq \sum_{l \in \mathbb{N}} \int_{T_{k,l}} |G(P)| \mathbb{1}_{\Delta E_{2^k}}(P) d\mathcal{L}(P)$$



$$\leq \|G\mathbb{1}_{\Delta E_{2^k}}\|_S \sum_{l \in \mathbb{N}} \bar{\mu}(T_{k,l}) \leq 2^{k+1} \mu(\Delta E_{2^k})$$

Thus

$$\sum_{k=-\infty}^{+\infty} \int_{\Delta E_{2^k}} |G(P)| d\mathcal{L}(P) \lesssim \|G\|_{L^1(\mathcal{X}, \mu, S)}.$$

The term  $\int_{\mathcal{X} \setminus E_0} |G(P)| d\mathcal{L}(P)$  vanishes because we may represent  $\mathcal{X} = \bigcup_{i \in \mathbb{N}} T_i$ . Using (2.45) and the monotonicity of sizes we have

$$\begin{aligned} \int_{\mathcal{X} \setminus E_0} |G(P)| d\mathcal{L}(P) &\leq \sum_{i \in \mathbb{N}} \int_{T_i} |G(P)| \mathbb{1}_{\mathcal{X} \setminus E_0}(P) d\mathcal{L}(P) \\ &\lesssim \sum_{i \in \mathbb{N}} \|G\mathbb{1}_{\mathcal{X} \setminus E_0}\|_{S(T_i)} \bar{\mu}(T_i) = 0. \end{aligned}$$

The term  $\int_{E_{+\infty}} |G(P)| d\mathcal{L}(P)$  also vanishes since

$$\mu(E_{2^k}) \leq \sum_{l=k}^{\infty} \mu(E'_{2^k}) \lesssim 2^{-k} \|G\|_{L^1(S)}$$

and thus  $\mu(E_{+\infty}) = 0$  and  $\mathcal{L}(E_{+\infty}) = 0$  by (2.46). This concludes the proof.  $\square$

The proof of the following outer measure Hölder inequality can be found in [DT15].

**Proposition 2.7** (Outer Hölder inequality). *Let  $(\mathcal{X}, \mu)$  be an outer measure space endowed with three sizes  $\|\cdot\|_S$ ,  $\|\cdot\|_{S'}$ , and  $\|\cdot\|_{S''}$  such that for any Borel functions  $F$  and  $A$  on  $\mathcal{X}$  the product estimate for sizes*

$$\|FA\|_S \lesssim \|F\|_{S'} \|A\|_{S''} \quad (2.48)$$

*holds. Then for any Borel functions  $F$  and  $A$  on  $\mathcal{X}$  the following outer Hölder inequality holds:*

$$\|FA\|_{L^p(S)} \leq 2 \|F\|_{L^{p'}(S')} \|A\|_{L^{p''}(S'')} \quad (2.49)$$

*for any triple  $p, p', p'' \in (0, \infty]$  of exponents such that  $\frac{1}{p'} + \frac{1}{p''} = \frac{1}{p}$ ,*

The above two statement can be easily extended to iterated outer measure spaces. Suppose from now on that  $\mathcal{X}$  is endowed with two outer measures  $\nu$  and  $\mu$ , the former generated by a pre-measure  $(\bar{\nu}, \mathbb{D})$  as described in (2.33). Given a size  $\|\cdot\|_S$  we introduce local  $L^q(S)$  sizes as described by (2.39) and the corresponding iterated outer  $L^p L^q(S)$  quasi-norms as described in (2.40).

**Corollary 2.8** (Outer Hölder inequality for iterated outer measure spaces). *Let  $(\mathcal{X}, \mu)$  be an outer measure space endowed with three sizes  $\|\cdot\|_S$ ,  $\|\cdot\|_{S'}$ , and  $\|\cdot\|_{S''}$  satisfying the assumptions of Proposition 2.7. Then given any two triples pairs of exponents  $p, p', p'' \in (0, \infty]$  and  $q, q', q'' \in (0, \infty]$  such that  $\frac{1}{p'} + \frac{1}{p''} = \frac{1}{p}$  and  $\frac{1}{q'} + \frac{1}{q''} = \frac{1}{q}$  the iterated Hölder bounds*

$$\|FA\|_{L^p L^q(S)} \lesssim \|F\|_{L^{p'} L^{q'}(S')} \|A\|_{L^{p''} L^{q''}(S'')} \quad (2.50)$$

*hold for any Borel functions  $F$  and  $A$  on  $\mathcal{X}$ .*

As a matter of fact the inequality

$$\|FA\|_{\mathcal{L}^q(S)} \lesssim \|F\|_{\mathcal{L}^{q'}(S')} \|A\|_{\mathcal{L}^{q''}(S'')} \quad (2.51)$$

holds for localized  $\mathcal{L}^q(S)$  sizes satisfy the inequality by Proposition 2.7 applied to the defining expression (2.39). Thus the local  $\mathcal{L}^q(S)$  sizes themselves satisfy the conditions of Hölder inequality and the statement of the above Corollary follows.

The Radon-Nikodym Lemma 2.6 can also be generalized to iterated outer measure  $L^p$  spaces.

**Corollary 2.9** (Iterated Radon-Nikodym domination). *Consider  $(\mathcal{X}, \mu)$  an outer measure space with a size  $\|\cdot\|_S$  and a Borel measure  $\mathcal{L}$  that satisfy the conditions of Lemma 2.6 and let  $\nu$  be a measure generated by  $(\bar{\nu}, \mathbb{D})$ . Suppose that  $\mathbb{D}$  also satisfies the covering condition of Lemma 2.6. Then the iterated Radon-Nikodym domination*

$$\left| \int_{\mathcal{X}} G(P) d\mathcal{L}(P) \right| \lesssim \|G(P)\|_{L^1 \mathcal{L}^1(S)}$$

holds.

As a matter of fact for any Borel function  $G$  the inequality

$$\int_D |G(P)| \mathcal{L}(P) \lesssim \|G\|_{\mathcal{L}^1(S)(D)} \bar{\nu}(D)$$

follows from (2.39) and Lemma 2.6. Thus the outer measure space  $(\mathcal{X}, \nu)$  and the family of local sizes  $\|\cdot\|_{\mathcal{L}^1(S)(D)}$  satisfy the conditions of Lemma 2.6 and the statement of the Corollary follows. Using the above properties one can deduce inequality (2.9): introduce the size

$$\|G\|_{S^1} := \sup_{T \in \mathbb{T}} \|G\|_{S^1(T)} = \sup_{T \in \mathbb{T}} \frac{\|G \mathbb{1}_T\|_{L^1}}{\mu(T)}$$

so that the sizes  $\|\cdot\|_{S^1}$ ,  $\|\cdot\|_{S_e}$ , and  $\|\cdot\|_{S_m}$  satisfy the product estimate (2.48). It follows from the iterated Hölder inequality (2.49) that

$$\|FA\|_{L^1 \mathcal{L}^1(S^1)} \lesssim \|F\|_{L^p \mathcal{L}^q(S_e)} \|A\|_{L^{p'} \mathcal{L}^{q'}(S_m)}$$

for conjugate exponents  $\frac{1}{p} + \frac{1}{p'} = 1$  and  $\frac{1}{q} + \frac{1}{q'} = 1$ . Furthermore we may apply 2.9 to  $(\mathcal{X}, \nu)$  with the local size  $\|\cdot\|_{\mathcal{L}^1(S^1)}$  so (2.9) follows.

### Interpolation

Here we recall some interpolation properties of outer measure  $L^p$  spaces from [DT15] and extend them to iterated outer measure  $L^p$  spaces.

The proof of the following Propositions can be found in [DT15].

**Proposition 2.10** (Logarithmic convexity of  $L^p$  norms). *Let  $(\mathcal{X}, \mu)$  be an outer measure space with size  $\|\cdot\|_S$  and let  $G$  be a Borel function on  $\mathcal{X}$ . For every  $\theta \in (0, 1)$  and for  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$  with  $p_0, p_1 \in (0, \infty]$ ,  $p_0 \neq p_1$  the inequality*

$$\|G\|_{L^{p_\theta}(S)} \leq C_{\theta, p_0, p_1} \|G\|_{L^{p_0, \infty}(S)}^{1-\theta} \|G\|_{L^{p_1, \infty}(S)}^\theta$$

holds.

The following straight-forward remarks are useful to be able to compare outer measure spaces with differing sizes.

**Remark 2.11** (Monotonicity of outer  $L^p$  spaces). *Consider an outer measure space  $(\mathbb{X}, \mu)$  with two sizes  $\|\cdot\|_S$  and  $\|\cdot\|_{S'}$ . Suppose that given two Borel functions  $G$  and  $G'$  on  $\mathbb{X}$  we have that*

$$\|G\mathbb{1}_{\mathbb{X}\setminus E}\|_S \lesssim \|G'\mathbb{1}_{\mathbb{X}\setminus E}\|_{S'}$$

for any  $E = \bigcup_{n \in \mathbb{N}} T_n$  that is countable union of generating sets  $T_n \in \mathbb{T}$ . Then

$$\|G\|_{L^p(S)} \lesssim \|G'\|_{L^p(S')}$$

for all  $p \in (0, \infty]$  and for iterated spaces

$$\|G\|_{L^p L^q(S)} \lesssim \|G'\|_{L^p L^q(S')}$$

for all  $p, q \in (0, \infty]$ . Similar statements hold for weak spaces.

**Remark 2.12** (Interpolation of sizes). *Given an outer measure space  $(\mathbb{X}, \mu)$  with two sizes  $\|\cdot\|_S$  and  $\|\cdot\|_{S'}$ , define the sum size as  $\|\cdot\|_{S+S'} := \|\cdot\|_S + \|\cdot\|_{S'}$ . Then the following inequality holds for any Borel function  $G$  and for any  $p \in (0, \infty]$*

$$\|G\|_{L^p(S+S')} \leq 2 (\|G\|_{L^p(S)} + \|G\|_{L^p(S')}).$$

The proofs of the above remarks consists of simply applying the definition of outer measure  $L^p$  quasi-norms and as such are left to the reader.

As a consequence of the above properties, given a function  $G$  the following inequality holds:

$$\|G\|_{L^p L^q(S)} \leq C_{q, q_0, q_1} (\|G\|_{L^p L^{q_0, \infty}(S)} + \|G\|_{L^p L^{q_1, \infty}(S)})$$

for all  $q_0, q_1 \in (1, \infty]$  and  $q \in (q_0, q_1)$ .

Finally we state a version of the Marcinkiewicz interpolation for maps into outer measure  $L^p$  spaces

**Proposition 2.13** (Marcinkiewicz interpolation). *Let  $(Y, \mathcal{L})$  be a classical measure space,  $(\mathbb{X}, \mu)$  be an outer measure space with size  $\|\cdot\|_S$  and assume  $1 \leq p_0 < p_1 \leq \infty$ . Let  $T$  an operator that maps  $L^{p_0}(Y, \mathcal{L}) + L^{p_1}(Y, \mathcal{L})$  to Borel function on  $\mathbb{X}$  so that*

**Scaling**  $|T(\lambda f)| = |\lambda T(f)|$  for all  $f \in L^{p_0}(Y, \mathcal{L}) + L^{p_1}(Y, \mathcal{L})$  and  $\lambda \in \mathbb{R}$ ;

**Quasi sub-additivity**

$$|T(f+g)| \leq C (|T(f)| + |T(g)|) \text{ for all } f, g \in L^{p_0}(Y, \mathcal{L}) + L^{p_1}(Y, \mathcal{L});$$

**Boundedness**

$$\begin{aligned} \|T(f)\|_{L^{p_0, \infty}(S)} &\leq C_1 \|f\|_{L^{p_0}(Y, \mathcal{L})} \quad \forall f \in L^{p_0}(Y, \mathcal{L}) \\ \|T(g)\|_{L^{p_1, \infty}(S)} &\leq C_2 \|g\|_{L^{p_1}(Y, \mathcal{L})} \quad \forall g \in L^{p_1}(Y, \mathcal{L}). \end{aligned}$$

Then for all  $f \in L^{p_0}(Y, \mathcal{L}) \cap L^{p_1}(Y, \mathcal{L})$  we have

$$\|T(f)\|_{L^{p_\theta}(S)} \lesssim_{\theta, p_0, p_1} C_1^{1-\theta} C_2^\theta \|f\|_{L^{p_\theta}(Y, \mathcal{L})}$$

with  $\theta \in [0, 1]$  and  $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ .

### 2.3 Wave packet decomposition

The main object of this section is to show inequality (2.13) i.e. the domination of the linearized variational Carleson operator via embedding maps. The following procedure follows the general scheme for obtaining (2.8) (2.3).

**Lemma 2.14.** *Consider any fixed parameters  $d > b > 0$ ,  $0 < d' < d - 2b$ ,  $d'' > d + 2b$ , and a small enough  $\varepsilon > 0$  appearing in properties (2.16), (2.17), and (2.18). There exists a choice of truncated left and right wave packets  $\Psi_{0,\eta,t}^{c_-,c_+,l}$  and  $\Psi_{0,\eta,t}^{c_-,c_+,r}$  such that for all  $c_- < c_+ \in \mathbb{R} \cup \{+\infty\}$  the expansion*

$$\mathbb{1}_{(c_-,c_+)}(\xi) = \iint_{\mathbb{R} \times \mathbb{R}^+} (\widehat{\Psi}_{0,\eta,t}^{c_-,c_+,l}(\xi) + \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,r}(\xi)) d\eta dt \quad (2.52)$$

holds where the integral converges in locally uniformly for  $\xi$  in  $(c_-, c_+)$ .

*Proof.* Let  $\varphi \in S(\mathbb{R})$  and  $\chi \in C_c^\infty(\mathbb{R})$  be two non-negative functions such that for  $\varepsilon > 0$  small enough, to be determined later the following holds

$$\text{spt } \widehat{\varphi} \subset B_b \quad \text{spt } \chi \subset B_\varepsilon(d) \subset (b, +\infty) \quad \iint_{\mathbb{R} \times \mathbb{R}^+} \widehat{\varphi}(\tilde{t} - \tilde{\eta}) \chi(\tilde{\eta}) d\tilde{\eta} \frac{d\tilde{t}}{\tilde{t}}. \quad (2.53)$$

A change of variable  $\tilde{\eta} = t\eta$  and  $\tilde{t} = \frac{t}{|\xi|}$ , gives:

$$\mathbb{1}_{(0,+\infty)}(\xi) = \iint_{\mathbb{R} \times \mathbb{R}^+} \widehat{\varphi}_{\eta,t}(\xi) \chi(t\eta) d\eta dt \quad \text{with } \varphi_{\eta,t}(z) := e^{i\eta z} t^{-1} \varphi\left(\frac{z}{t}\right). \quad (2.54)$$

Let  $\gamma \in C_c^\infty([0, 1 + \varepsilon])$  so that

$$\gamma(t) = 1 \text{ for } t \in [0, (1 + \varepsilon)^{-1}] \quad \gamma(t) + \theta(1/t) = 1 \text{ for } t \in \mathbb{R}^+.$$

Such a function can be constructed by taking  $\tilde{\gamma}$  to satisfy the first two conditions and by setting  $\gamma(t) := \frac{\tilde{\gamma}(t)}{\tilde{\gamma}(t) + \tilde{\gamma}(1/t)}$ . Let us then set

$$\begin{aligned} \beta(\xi) &:= \iint_{\mathbb{R} \times \mathbb{R}^+} \gamma(t') \widehat{\varphi}_{\eta',t'}(\xi) \chi(t'\eta') d\eta' dt' \quad \text{so that} \\ \beta(t\xi) &= \iint_{\mathbb{R} \times \mathbb{R}^+} \gamma(t'/t) \widehat{\varphi}_{\eta',t'}(\xi) \chi(t'\eta') d\eta' dt'. \end{aligned} \quad (2.55)$$

Using (2.54) one obtains

$$\mathbb{1}_{(c_-,c_+)}(\xi) = \iiint_{(\mathbb{R} \times \mathbb{R}^+)^2} \widehat{\varphi}_{\eta,t}(\xi) \chi(t(\eta - c_-)) \widehat{\varphi}_{\eta',t'}(\xi) \chi(t'(c_+ - \eta')) d\eta' dt' d\eta dt,$$

so the representation (2.52) holds with

$$\begin{aligned} \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,l}(\xi) &:= \chi(t(\eta - c_-)) \widehat{\varphi}_{\eta,t}(\xi) \beta(t(c_+ - \xi)) \\ \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,r}(\xi) &:= \chi(t(c_+ - \eta)) \widehat{\varphi}_{\eta,t}(\xi) \beta(t(\xi - c_-)). \end{aligned} \quad (2.56)$$

It remains to check that  $\Psi_{0,\eta,t}^{c_-,c_+,l}$  are left truncated wave packets. By symmetry it will follow that  $\Psi_{0,\eta,t}^{c_-,c_+,r}$  is a right truncated wave packet. First of all (2.16) holds according to (2.56) since  $\text{spt } \widehat{\varphi}_{\eta,t}(\xi) \subset B_{bt^{-1}}(\eta)$ .

Notice that

$$\text{spt } \beta \subset \left( \frac{d - \varepsilon - b}{1 + \varepsilon}, +\infty \right) \quad \beta(\xi) = 1 \text{ on } ((d + \varepsilon + b)(1 + \varepsilon), +\infty). \quad (2.57)$$

As a matter of fact the integrand in (2.55) is non-zero only if  $t'(\xi - \eta') \in B_b$  and  $t'\eta' \in B_\varepsilon(d)$  so  $t'\xi \in B_{\varepsilon+b}(d)$ . This shows that

$$\begin{aligned} \xi \leq \frac{d - \varepsilon - b}{1 + \varepsilon} &\implies t' > 1 + \varepsilon \text{ or } t' < 0 \implies \gamma(t') = 0 \implies \beta(\xi) = 0 \\ \xi \geq (d + \varepsilon + b)(1 + \varepsilon) &\implies t' < (1 + \varepsilon)^{-1} \implies \gamma(t') = 1 \implies \beta(\xi) = 1 \end{aligned}$$

where the last equality follows from (2.54).

We now check that (2.17) holds. It follows from (2.56) that  $\Psi_{y,\eta,t}^{c_-,c_+,l}(\xi)$  vanishes unless  $\chi(t(\eta - c_-)) \neq 0$  i.e. unless  $t(\eta - c_-) \in B_\varepsilon(d)$ . Also  $\Psi_{y,\eta,t}^{c_-,c_+,l}(\xi) = 0$  unless  $t(\xi - \eta) > -b$  and  $t(c_+ - \xi) > \frac{d - \varepsilon - b}{1 + \varepsilon}$  i.e. unless  $t(c_+ - \eta) > \frac{d - \varepsilon - b}{1 + \varepsilon} - b$ . As long as  $0 < d' < d - 2b$  one can choose  $\varepsilon > 0$  small enough for (2.17) to hold.

We now check that (2.18) holds. We have that  $\beta(t(c_+ - \xi)) = 1$  if  $t(c_+ - \xi) > (d + \varepsilon + b)(1 + \varepsilon)$  and we know that  $\widehat{\varphi}_{\eta,t}(\xi) \neq 0$  only if  $t(\xi - \eta) \in B_b$  thus if  $t(c_+ - \eta) > (d + \varepsilon + b)(1 + \varepsilon) + b$  then

$$\widehat{\Psi}_{0,\eta,t}^{c_-,c_+,l} = \chi(t(\eta - c_-)) \widehat{\varphi}_{\eta,t}(\xi) =: \widehat{\Psi}_{0,\eta,t}^{c_-,+\infty,l}$$

so (2.18) holds as long as  $d'' > d + 2b$  and  $\varepsilon > 0$  is chosen small enough

We now need to check the smoothness conditions (2.15). We must show that the functions

$$\widehat{\Psi}_{0,\eta,t}^{c_-,c_+,l} \left( \frac{\xi + \eta}{t} \right) \quad t^{-1} \partial_{c_-} \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,r} \left( \frac{\xi + \eta}{t} \right) \quad t^{-1} \partial_{c_+} \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,r} \left( \frac{\xi + \eta}{t} \right)$$

are all uniformly bounded in  $S(\mathbb{R})$  for all  $\eta, t \in \mathbb{R} \times \mathbb{R}^+$  and  $c_- < c_+ \in \mathbb{R}$ . Clearly

$$\widehat{\Psi}_{0,\eta,t}^{c_-,c_+,l} \left( \frac{\xi + \eta}{t} \right) = \chi(t\eta - tc_-) \widehat{\varphi}_{\eta,t} \left( \frac{\xi + \eta}{t} \right) \beta(tc_+ - \xi + \eta)$$

and the claim follows.  $\square$

**Corollary 2.15.** *Let us fix a set of parameters  $d'', d', d > 0$  with  $d'' > \max(d', d)$  and  $3d > d'$ . Then for any  $\varepsilon > 0$  small enough there exists  $b > 0$  such that there exists a choice of left and right truncated wave packets  $\Psi_{0,\eta,t}^{c_-,c_+,l}$  and  $\Psi_{0,\eta,t}^{c_-,c_+,r}$  such that (2.52) holds for all  $c_- < c_+ \in \mathbb{R} \cup \{+\infty\}$ .*

*Proof.* If  $d'' > d > d' > 0$  then let us choose  $\varepsilon > 0$  and  $b > 0$  small enough so that the conditions for Lemma 2.14 hold. Then the Lemma provides us with wave packets  $\Psi_{0,\eta,t}^{c_-,c_+,l}$  and  $\Psi_{0,\eta,t}^{c_-,c_+,r}$  such that (2.52) holds as required.

Suppose now that  $3d > d' \geq d$  and  $d'' > d'$  and consider the set of parameters  $\widetilde{d}'', \widetilde{d}', \widetilde{d}, \widetilde{b}, \widetilde{\varepsilon} > 0$  given by

$$\widetilde{\varepsilon} = \varepsilon \quad \widetilde{b} = b - \delta \quad \widetilde{d} = d + \delta \quad \widetilde{d}' = d' - \delta \quad \widetilde{d}'' = d'' - \delta$$

for some  $d > \delta > 0$ . We need to check that the above parameters satisfy the assumptions of Lemma 2.14 that will give us the left and right truncated wave-packets  $\widetilde{\Psi}_{0,\eta,t}^{c_-,c_+,l}$  and  $\widetilde{\Psi}_{0,\eta,t}^{c_-,c_+,r}$  for which (2.16), (2.17), and (2.18) hold with these modified parameters. As long as  $2\delta + \widetilde{b} < \widetilde{d} - \widetilde{\varepsilon}$ , setting  $\Psi_{0,\eta,t}^{c_-,c_+,l} := \widetilde{\Psi}_{0,\eta+\delta t^{-1},t}^{c_-,c_+,l}$  and  $\Psi_{0,\eta,t}^{c_-,c_+,r} := \widetilde{\Psi}_{0,\eta-\delta t^{-1},t}^{c_-,c_+,r}$  will provide us with the required wave-packets so that (2.52) holds.

Set  $b = \frac{d'-d}{2(1-3\varepsilon)}$  and  $\delta = (1-\varepsilon)b$  so that  $\tilde{b} = \varepsilon b$  with  $\varepsilon > 0$  small enough for the subsequent inequalities to hold. All the abovementioned conditions hold since

$$\begin{aligned}\tilde{d} - \varepsilon - \tilde{b} - 2\delta &= d + \delta - \varepsilon - b + \delta - 2\delta = d - \varepsilon - b = d - \varepsilon - \frac{d' - d}{2(1 - 3\varepsilon)} > 0 \\ \tilde{b} &= \varepsilon b > 0 \\ \tilde{d} - \tilde{b} &= d - b + 2\delta > 0 \\ \tilde{d}' &> d' - \delta = d' - \frac{1 - \varepsilon}{2} \frac{d' - d}{1 - 3\varepsilon} > 0 \\ \tilde{d} - 2\tilde{b} - \tilde{d}' &= d - d' - 2b + 4\delta = d - d' + 2(1 - 2\varepsilon)b = (d' - d) \left( \frac{1 - 2\varepsilon}{1 - 3\varepsilon} - 1 \right) > 0 \\ \tilde{d}'' - \tilde{d} - 2\tilde{b} &= d'' - d - 2b > d'' - d' + (d' - d) \left( 1 - \frac{1}{1 - 3\varepsilon} \right) > 0.\end{aligned}$$

This concludes the proof.  $\square$

As a consequence we obtain the following representation Lemma.

**Lemma 2.16.** *Let us fix a set of parameters  $d'', d', d > 0$  with  $d'' > \max(d', d)$  and  $3d > d'$ . For any  $\varepsilon > 0$  small enough there exists  $b > 0$  such that for any  $f \in S(\mathbb{R})$  and  $c_- < c_+ \in \mathbb{R} \cup \{+\infty\}$  the expansion*

$$\int_{c_-}^{c_+} \widehat{f}(\xi) e^{i\xi x} d\xi = \iiint_{\mathcal{X}} f * \psi_{\eta,t}(y) \left( \Psi_{y,\eta,t}^{c_-,c_+,l}(z) + \Psi_{y,\eta,t}^{c_-,c_+,r}(z) \right) dy d\eta dt \quad (2.58)$$

holds. Here  $\Psi_{y,\eta,t}^{c_-,c_+,l}$  and  $\Psi_{y,\eta,t}^{c_-,c_+,r}$  are some left and right truncated wave packets for which properties (2.15), (2.16), (2.17), and (2.18) hold with the parameters above. The function  $\psi_{\eta,t}$  is obtained from some  $\psi \in S(\mathbb{R})$  as in (2.4); we also have

$$\text{spt } \widehat{\psi} \in B_{(1+\varepsilon)b} \quad \text{with } (1+\varepsilon)b < d - \varepsilon.$$

*Proof.* Let us choose  $\psi \in S(\mathbb{R})$  such that  $\text{spt } \widehat{\psi} \in B_{(1+\varepsilon)b}$  and  $\widehat{\psi} = 1$  on  $B_b$  so that

$$\widehat{\Psi}_{0,\eta,t}^{c_-,c_+,l}(\xi) = \widehat{\psi}_{\eta,t}(\xi) \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,l}(\xi) \quad \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,r}(\xi) = \widehat{\psi}_{\eta,t}(\xi) \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,r}(\xi)$$

and let us set  $\Psi_{y,\eta,t}^{c_-,c_+,l}(z) = \Psi_{0,\eta,t}^{c_-,c_+,l}(z - y)$  and  $\Psi_{y,\eta,t}^{c_-,c_+,r}(z) = \Psi_{0,\eta,t}^{c_-,c_+,r}(z - y)$ . It follows that

$$\begin{aligned}& \iiint_{\mathcal{X}} f * \psi_{\eta,t}(y) \left( \Psi_{y,\eta,t}^{c_-,c_+,l}(z) + \Psi_{y,\eta,t}^{c_-,c_+,r}(z) \right) dy d\eta dt \\ &= \iint_{\mathbb{R} \times \mathbb{R}^+} f * \psi_{\eta,t}(y) * \left( \Psi_{0,\eta,t}^{c_-,c_+,l} + \Psi_{0,\eta,t}^{c_-,c_+,r} \right)(z) d\eta dt \\ &= \mathcal{F}^{-1} \left( \iint_{\mathbb{R} \times \mathbb{R}} \widehat{f}(\xi) \widehat{\psi}_{\eta,t}(\xi) \left( \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,l}(\xi) + \widehat{\Psi}_{0,\eta,t}^{c_-,c_+,r}(\xi) \right) \right) = \mathcal{F}^{-1} \left( \widehat{f}(\xi) \mathbb{1}_{c_-,c_+}(\xi) \right)\end{aligned}$$

as required, where  $\mathcal{F}^{-1}$  is the inverse Fourier transform.  $\square$

As a corollary of the above Lemma we have the following pointwise wave-packet representation for the linearized variational Carleson operator:

$$\sum_{k \in \mathbb{Z}} a_k(z) \int_{c_k(z)}^{c_{k+1}(z)} \widehat{f}(\xi) e^{i\xi z} d\xi$$

$$= \sum_{k \in \mathbb{Z}} \iiint_{\mathbb{X}} f * \psi_{\eta,t}(y) \left( \Psi_{y,\eta,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z), l}(z) + \Psi_{y,\eta,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z), r}(z) \right) a_k(z) d\eta dy dt.$$

Setting

$$\mathbb{A}_{\mathbf{c}}(y, \eta, t) := \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \left( \Psi_{y,\eta,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z), l}(z) + \Psi_{y,\eta,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z), r}(z) \right) a_k(z) dz$$

gives (2.22). We also remark that if  $\mathbf{c}$  and  $\mathbf{a}$  are as in (2.21) then the above construction reduces to the one described by (2.3), (2.6) and (2.7) thus showing (2.23).

Finally notice that if we fix  $\Theta = (\alpha^-, \alpha^+) = (-1, 1)$  and set  $d < 1 < d' < d''$  with  $d' < 3d$ , then for every  $\varepsilon > 0$  small enough we may apply Lemma 2.16 to obtain the parameter  $b > 0$  and wave-packets  $\Psi_{y,\eta,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z), l}(z)$ ,  $\Psi_{y,\eta,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z), r}(z)$ , and  $\psi_{\eta,t}$ . Supposing that  $\varepsilon > 0$  is small enough so that  $d + \varepsilon < \alpha^+ = 1$  we can find  $(1 + \varepsilon)b < \beta^+ < d - \varepsilon$  and set  $\Theta^{(i)} = (\beta^-, \beta^+) = (-\beta^+, \beta^+)$ . Thus there exists a set of parameters  $\alpha^- < \beta^- < \beta^+ < \alpha^+$  such that (2.13) holds and (2.29) is satisfied so that Theorem 2.3 and Theorem 2.2 hold.

## 2.4 The auxiliary embedding map

In this section we introduce an auxiliary embedding map used to control the embedded function  $\mathbb{A}$ . The bounds with the same exponents as in (2.19) hold for the auxiliary embedded function  $\mathbb{M}$  with  $S^\infty$  in lieu of  $S_m$ . However it is technically easier to control the super-level outer measure  $\mu(\|\mathbb{M}\|_{S^\infty} > \lambda)$  of the auxiliary embedded function  $\mathbb{M}$ . A crucial covering Lemma implies non-iterated outer  $L^{p'}$  space bounds for  $\mathbb{M}$  while a locality property and a projection Lemma allows for the extension to iterated outer  $L^{p'}E^q$  spaces.

The auxiliary embedding map associates to  $\mathbf{a} \in C_c^\infty(I^{r'})$  the function on  $\mathbb{X}$  given by

$$\mathbb{M}(y, \eta, t) := \int_{\mathbb{R}} \left( \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_\Theta(t(\eta - \mathbf{c}_k(z))) \right)^{1/r'} W_t(z - y) dz \quad (2.59)$$

where the bump function  $W$  is as in (2.24).

**Proposition 2.17** (Bounds on the auxiliary embedding map  $\mathbb{M}$ ). *For any  $r' \in [1, \infty]$ ,  $p' \in (1, \infty]$ , and  $q' \in (r', \infty]$  and for any function  $\mathbf{a} \in L^{p'}(I^{r'})$  the function  $\mathbb{M}$  defined by (2.59) satisfies the bounds*

$$\|\mathbb{M}\|_{L^{p'}E^{q'}(S^\infty)} \lesssim \|\mathbf{a}\|_{L^{p'}(I^{r'})} \quad (2.60)$$

where  $S^\infty(\mathbb{M}) := \sup_{(y,\eta,t) \in \mathbb{X}} \mathbb{M}(y, \eta, t)$ . Furthermore the weak endpoint bounds

$$\begin{aligned} \|\mathbb{M}\|_{L^{p'}E^{r'}, \infty(S^\infty)} &\lesssim \|\mathbf{a}\|_{L^{p'}(I^{r'})} & p' \in (1, \infty] \\ \|\mathbb{M}\|_{L^{1, \infty}E^{q'}(S^\infty)} &\lesssim \|\mathbf{a}\|_{L^1(I^{q'})} & q' \in (r', \infty] \\ \|\mathbb{M}\|_{L^{1, \infty}E^{r'}, \infty(S^\infty)} &\lesssim \|\mathbf{a}\|_{L^1(I^{r'})}. \end{aligned} \quad (2.61)$$

hold. All the above inequalities hold as long as  $N > 0$  in (2.24) is large enough and with constants independent of the stopping sequence  $\mathbf{c}$  appearing in (2.59).

We may make two reductions to prove the above bounds. First of all one can substitute  $W_t(z)$  by a normalized characteristic function of a ball. As a matter of fact set

$$\mathbb{M}_R(y, \eta, t) := \int_{B_{Rt}(y)} \left( \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_\Theta(t(\eta - \mathbf{c}_k(z))) \right)^{1/r'} dz$$

so that  $\mathbb{M}(y, \eta, t) \lesssim \sum_{n \in \mathbb{N}} R^{-Nn} \mathbb{M}_{R^n}(y, \eta, t)$ . Thus it is sufficient to prove that the bounds (2.60) hold for  $\mathbb{M}_R$  with a constant that grows at most as  $R^{N'}$  for some  $N' > 0$  as  $R \rightarrow \infty$ . The bounds for  $\mathbb{M}$  follow by quasi-subadditivity as remarked in Section 2.2.1 as long as  $N > N'$ . For the second reduction split  $\Theta = \Theta^+ \cup \Theta^-$  into  $\Theta^+ := \Theta \cap [0, +\infty]$  and  $\Theta^- := \Theta \cap [-\infty, 0]$ . Set

$$\mathbb{M}_R^\pm(y, \eta, t) := \int_{B_{Rt}(y)} \left( \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_{\Theta^\pm}(t(\eta - \mathbf{c}_k(z))) \right)^{1/r'} dz \quad (2.62)$$

so that  $\mathbb{M}_R \leq \mathbb{M}_R^+ + \mathbb{M}_R^-$ . Thus it will suffice to provide the proof of the bounds (2.60) only for  $\mathbb{M}_R^+$ .

We begin by introducing the concept of disjoint tents relative to the embedding (2.62) and record an important covering lemma.

**Definition 2.18** ( $Q^+$ -disjointness). *Let  $Q > 0$ . We say two tents  $T(x, \xi, s)$  and  $T(x', \xi', s')$  are  $Q^+$ -disjoint if either*

$$B_{Qs}(x) \cap B_{Qs'}(x') = \emptyset \quad \text{or} \quad \{c: s(\xi - c) \in \Theta^+\} \cap \{c: s'(\xi' - c) \in \Theta^+\} = \emptyset.$$

Notice that if a sequence of tents  $T(x_l, \xi_l, s_l)_{l \in \mathbb{N}}$  are pairwise  $Q^+$ -disjoint, with  $Q \geq R$ , then for every  $z \in \mathbb{R}$

$$\left| \sum_{l \in \mathbb{N}} \mathbb{1}_{\Theta^+}(s_l(\xi_l - \mathbf{c}_k(z))) \mathbb{1}_{B_R}\left(\frac{x_l - z}{s_l}\right) \right| \leq 1$$

and the bound

$$\begin{aligned} \sum_{l \in \mathbb{N}} s_l \mathbb{M}_R^+(x_l, \xi_l, s_l)^{r'} &\leq \sum_{l \in \mathbb{N}} s_l \int_{B_{R s_l}(x_l)} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_{\Theta^+}(t(\xi_l - \mathbf{c}_k(z))) dz \\ &\leq (2R)^{-1} \int_{\mathbb{R}} \|\mathbf{a}(z)\|_{l^{r'}}^{r'} dz = (2R)^{-1} \|\mathbf{a}\|_{L^{r'}(l^{r'})}^{r'} \end{aligned} \quad (2.63)$$

holds.

What follows is a covering lemma. We remark that this is the only instance where we require smoothness and rapid decay assumptions on  $\mathbf{a}$ .

**Lemma 2.19.** *Let  $\mathbf{a} \in C_c^\infty(l^{r'})$ . If  $Q > R > R_0$  for some  $R_0 > 0$  depending on  $\Theta$  the super level set*

$$E_{\lambda, R} := \{(x, \xi, s): \mathbb{M}_R^+(x, \xi, s) \geq \lambda\}$$

*admits a finite covering  $\bigcup_{l=1}^L 3Q^2 T_l \supset E_{\lambda, R}$  with tents  $Q^+$ -disjoint tents  $T_l = T(x_l, \xi_l, s_l)$  centered at points  $(x_l, \xi_l, s_l) \in E_{\lambda, R}$ .*

*Proof.* Introduce the relation  $\triangleleft$  between points of  $\mathbb{X}$  such that  $(x, \xi, s) \triangleleft (x', \xi', s')$  if  $B_{Qs}(x) \cap B_{Qs'}(x') \neq \emptyset$ ,  $s(\xi - \xi') \in \Theta$  and  $s' > Qs$ . We say  $(x, \xi, s)$  is maximal in a set  $P \subset \mathbb{X}$  if there is no  $(x', \xi', s') \in P$  such that  $(x, \xi, s) \triangleleft (x', \xi', s')$ . Notice that  $E_{\lambda, R}$  is  $(x, t)$ -bounded in the sense that for some  $C > 1$  large enough

$$E_{\lambda, R} \subset B_C(0) \times \mathbb{R} \times (0, C)$$

holds. As a matter of fact  $\mathbb{M}_R^+(y, \eta, t) \lesssim (Rt)^{-1} \|\mathbf{a}\|_{L^1(l^{r'})}$  and  $\mathbb{M}_R^+(y, \eta, t) = 0$  if  $\text{dist}(y; \text{spt } \mathbf{a}) > tR$  so if  $(y, \eta, t) \in E_{\lambda, R}$  then  $t < C$  and  $|y| < C$  for some  $C > 0$  depending on  $\mathbf{a}$ . Thus any non-empty subset  $P \subset E_{\lambda, R}$  admits a maximal element.



Inductively construct a covering starting with an empty collection of tents  $\mathcal{T}^0 = \emptyset$ . At the  $l^{\text{th}}$  step consider the points in the set

$$E_{\lambda,R} \setminus \bigcup_{T \in \mathcal{T}^{l-1}} 3Q^2 T \quad (2.64)$$

and select from it a point  $(x_l, \xi_l, s_l)$  that is maximal with respect to the relation  $\triangleleft$  and set  $\mathcal{T}^l = \mathcal{T}^{l-1} \cup \{T(x_l, \xi_l, s_l)\}$ . We claim that at each step of the algorithm all the selected tent  $T(x_l, \xi_l, s_l)$  are pairwise  $Q^+$ -disjoint. Reasoning by contradiction, suppose that two tents  $T(x_l, \xi_l, s_l)$  and  $T(x_{l'}, \xi_{l'}, s_{l'})$  with  $l < l'$  are not  $Q^+$ -disjoint, then  $B_{Qs_l}(x_l) \cap B_{Qs_{l'}}(x_{l'}) \neq \emptyset$  and there also exists a  $c \in \mathbb{R}$  such that  $s_l(\xi_l - c) \in \Theta^+$  and  $s_{l'}(\xi_{l'} - c) \in \Theta^+$ . Recall that  $\Theta^+ = [0, \alpha^+]$  so

$$\xi_l - s_l^{-1}\alpha^+ \leq c \leq \xi_l \quad \xi_{l'} - s_{l'}^{-1}\alpha^+ \leq c \leq \xi_{l'}$$

If  $s_{l'} \geq Qs_l$  one would have

$$-s_l^{-1}Q^{-1}\alpha^+ \leq -s_{l'}^{-1}\alpha^+ \leq \xi_l - \xi_{l'} \leq s_l^{-1}\alpha^+$$

and thus  $s_l(\xi_l - \xi_{l'}) \in \Theta$  as long as  $\alpha^- \leq -R_0^{-1}\alpha^+$ . This contradicts the maximality of  $(x_l, \xi_l, s_l)$  that was chosen before  $(x_{l'}, \xi_{l'}, s_{l'})$ . On the other hand if  $s_{l'} < Qs_l$  then

$$-s_l^{-1}\alpha^+ \leq \xi_{l'} - \xi_l \leq s_{l'}^{-1}\alpha^+$$

and, as long as  $Q \geq R_0 \geq \alpha^+$ , this implies that  $(x_{l'}, \xi_{l'}, s_{l'}) \in 3Q^2T(x_l, \xi_l, s_l)$  contradicting the selection condition.

Finally notice that the selection algorithm terminates after finitely many steps since at every step (2.63) holds having chosen  $Q \geq R$ , since  $s_l$  are bounded from below since  $\mathbb{M}_R^+(x_l, \xi_l, s_l) \geq \lambda$ . Thus  $E_\lambda \subset \bigcup_{l=1}^L 3Q^2T_l$ .  $\square$

A consequence of the above Lemma are non-iterated bounds for  $\mathbb{M}_R^+$ .

**Proposition 2.20.** *Given  $\mathbf{a} \in L^{p'}(l^{r'})$  with  $p' \in (r', \infty]$  the bound*

$$\|\mathbb{M}_R^+\|_{L^{p'}(S^\infty)} \lesssim_R \|\mathbf{a}\|_{L^{p'}(l^{r'})} \quad (2.65)$$

*holds. Furthermore the weak endpoint bound*

$$\|\mathbb{M}_R^+\|_{L^{r',\infty}(S^\infty)} \lesssim_R \|\mathbf{a}\|_{L^{r'}(l^{r'})} \quad (2.66)$$

*holds. All the above bounds hold with a constant that grows at most polynomially in  $R$  as  $R \rightarrow \infty$  and is independent of the stopping sequence  $\mathbf{c}$  appearing in (2.62).*

The bound (2.65) for  $p = \infty$  is straightforward:

$$\begin{aligned} \mathbb{M}_R^+(y, \eta, t) &= \int_{B_{tR}(y)} \left( \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_{\Theta^+}(t(\eta - c_k(z))) \right)^{1/r'} dz \\ &\leq \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} t^{-1} \mathbb{1}_{B_R} \left( \frac{z-y}{t} \right) dz \lesssim \|\mathbf{a}\|_{L^\infty(l^{r'})}^{r'}. \end{aligned}$$

It is sufficient to show bound (2.66) so that will (2.65) follow for  $p \in (r', \infty)$  by interpolation 2.13. In particular to obtain (2.66) we will show that given  $\lambda > 0$  the bound on the measure of the super-level set

$$\mu(E_{\lambda,R}) \lesssim_R \lambda^{-r'} \|\mathbf{a}\|_{L^{r'}(l^{r'})}^{r'} \quad (2.67)$$

holds. It is sufficient to consider the covering provided by Lemma 2.19 with  $Q = R$ . Since  $(x_l, \xi_l, s_l) \in E_{\lambda, R}$  and the covering  $\mathcal{T} = \mathcal{T}^L = \{T(x_l, \xi_l, s_l)\}_{l \in L}$  consists of  $Q^+$ -disjoint tents, the bound (2.63) gives

$$\lambda^{r'} \sum_{l=1}^L s_l \leq (2R)^{-1} \|\mathbf{a}\|_{L^{r'}(I^{r'})}^{r'}.$$

Since  $E_{\lambda, R} \subset \bigcup_{l=1}^L 3R^3 T(x_l, \xi_l, s_l)$  one deduces

$$\mu(E_{\lambda, R}) \lesssim_R \sum_{l=1}^L \mu(T(x_l, \xi_l, s_l)) \leq \sum_{l=1}^L s_l \leq \frac{\|\mathbf{a}\|_{L^{r'}(I^{r'})}^{r'}}{\lambda^{r'}}$$

where the implied constant grows polynomially in  $R$  as required.

The proof of 2.17 relies on a locality property and a strip projection lemma.

**Lemma 2.21** (Locality of  $\mathbb{M}_R^+$ ). *Consider a strip  $D = D(x, s)$  and a function  $\mathbf{a} \in L_{loc}^1(I^{r'})$  with*

$$\text{dist}(\text{spt } \mathbf{a}; B_s(x)) > Rs$$

*then for all  $(y, \eta, t) \in D(x, s)$  we have*

$$\mathbb{M}_R^+ \mathbb{1}_{D(x, s)} = 0.$$

*Proof.* The statement follows directly from the definition (2.62) of the embedding. As a matter of fact if  $(y, \eta, t) \in D_{x, s}$  then  $B_{tR}(y) \subset B_{sR}(y)$  and  $B_{sR}(y) \cap \text{spt } \mathbf{a} = \emptyset$ .  $\square$

**Lemma 2.22** (Mass projection for  $\mathbb{M}_R^+$ ). *Fix any collection of pairwise disjoint strips  $D(\zeta_m, \tau_m)$ ,  $m \in \{1, \dots, M\}$  and any finite collection of  $Q^+$ -disjoint tents*

$$T(x_l, \xi_l, s_l) \not\subset \bigcup_{m=1}^M D(\zeta_m, 3\tau_m), \quad l \in \{1, \dots, L\}$$

*with  $Q > 2R > 2$ . Given a function  $\mathbf{a} \in L_{loc}^1(I^{r'})$  and a stopping sequence  $\mathbf{c}$  there exists a function  $\tilde{\mathbf{a}} \in L_{loc}^1(I^{r'})$  and a new stopping sequence  $\tilde{\mathbf{c}}$  such that*

$$\|\tilde{\mathbf{a}}(z)\|_{I^{r'}} \lesssim \int_{B_{\tau_m}(\zeta_m)} \|\mathbf{a}(z)\|_{I^{r'}} dz \quad \forall z \in B_{\tau_m}(\zeta_m) \quad \forall m \in \{1, \dots, M\} \quad (2.68)$$

$$\tilde{\mathbf{a}}_k(z) = \mathbf{a}_k(z) \quad \forall z \notin \bigcup_{m=1}^M B_{\tau_m}(\zeta_m)$$

and

$$\tilde{\mathbb{M}}_{2R}^+(x_l, \xi_l, s_l) \geq \mathbb{M}_R^+(x_l, \xi_l, s_l) \quad \forall l \in \{1, \dots, L\}. \quad (2.69)$$

where  $\tilde{\mathbb{M}}_{2R}^+$  is the embedded function as given by expression (2.62) associated to  $\tilde{\mathbf{a}}$  with the stopping sequence  $\tilde{\mathbf{c}}$ .

*Proof.* Let us order the tents  $T(x_l, \xi_l, s_l)$  so that  $\xi_l \leq \xi_{l'}$  if  $l < l'$ . For every strip  $D(\zeta_m, \tau_m)$  let

$$\mathbb{L}_m := \left\{ l \in \{1, \dots, L\} : D(x_l, Rs_l) \cap D(\zeta_m, \tau_m) \neq \emptyset \right\}.$$

Set

$$\tilde{\mathbf{a}}_k(z) = \begin{cases} \mathbf{a}_k(z) & \text{if } z \notin \bigcup_m B_{\tau_m}(\zeta_m) \\ \int_{B_{\tau_m}(\zeta_m)} \left( \sum_{j \in \mathbb{Z}} |\mathbf{a}_j(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_k - \mathbf{c}_j(z))) \right)^{1/r'} dz & \text{if } z \in B_{\tau_m}(\zeta_m) \\ & \text{and } k \in \mathbb{L}_m \\ 0 & \text{if } z \in B_{\tau_m}(\zeta_m) \\ & \text{and } k \notin \mathbb{L}_m \end{cases}$$

$$\tilde{\mathbf{c}}_k(z) = \begin{cases} \mathbf{c}_k(z) & \text{if } z \notin \bigcup_m B_{\tau_m}(\zeta_m) \\ \xi_k & \text{if } z \in B_{\tau_m}(\zeta_m) \text{ and } k \in \{1, \dots, L\} \\ \xi_1 & k < 1 \\ \xi_L & k > L. \end{cases}$$

The expressions above are well defined since  $D(\zeta_m, \tau_m)$  are pairwise disjoint.

The bound (2.68) follows by the Minkowski inequality. For  $z \in B_{\tau_m}(\zeta_m)$  one has

$$\begin{aligned} \|\tilde{\mathbf{a}}(z)\|_{l^{r'}} &= \left( \sum_{k \in \mathbb{Z}} \left( \int_{B_{\tau_m}(\zeta_m)} \left( \sum_{j \in \mathbb{Z}} |\mathbf{a}_j(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_k - \mathbf{c}_j(z))) \right)^{1/r'} dz \right)^{r'} \right)^{1/r'} \\ &\lesssim \int_{B_{\tau_m}(\zeta_m)} \left( \sum_{k \in \mathbb{L}_m} \sum_{j \in \mathbb{Z}} |\mathbf{a}_j(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_k - \mathbf{c}_j(z))) \right)^{1/r'} dz \leq \int_{B_{\tau_m}(\zeta_m)} \|\mathbf{a}(z)\|_{l^{r'}} dz, \end{aligned}$$

where the last inequality holds since the tents  $T(x_l, \xi_l, s_l)$  are  $Q^+$ -disjoint.

It remains to show (2.69). Since  $T(x_l, \xi_l, s_l) \not\subset D(\zeta_m, 3\tau_m)$  for any  $m$  we have that

$$B_{R s_l}(x_l) \cap B_{\tau_m}(\zeta_m) \neq \emptyset \implies D(\zeta_m, \tau_m) \subset D(x_l, 2R s_l)$$

so set

$$\mathfrak{M}_l = \left\{ m : D(\zeta_m, \tau_m) \subset D(x_l, 2R s_l) \right\}.$$

Using the definitions of  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{c}}$  we obtain

$$\begin{aligned} \tilde{\mathfrak{M}}_{2R}^+(x_l, \xi_l, s_l) &= \int_{B_{2R s_l}(x_l)} \left( \sum_{k \in \mathbb{Z}} |\tilde{\mathbf{a}}_k(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_l - \tilde{\mathbf{c}}_k(z))) \right)^{1/r'} dz \\ &\gtrsim (4R s_l)^{-1} \int_{B_{R s_l}(x_l) \setminus \bigcup_m B_{\tau_m}(\zeta_m)} \left( \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_l - \mathbf{c}_k(z))) \right)^{1/r'} dz \\ &\quad + (4R s_l)^{-1} \sum_{m \in \mathfrak{M}_l} \int_{B_{\tau_m}(\zeta_m)} \left( \sum_{k \in \mathbb{L}_m} |\tilde{\mathbf{a}}_k(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_l - \xi_k)) \right)^{1/r'} dz. \end{aligned}$$

Using the fact that  $T(x_l, \xi_l, s_l)$  are  $Q^+$ -disjoint with  $Q > 2R$  we obtain that  $s_l(\xi_l - \xi_k) \in \Theta^+$ ,  $z \in B_{\tau_m}(\zeta_m)$ , and  $\tilde{\mathbf{a}}_k(z) \neq 0$  only if  $k = l$ ; thus

$$\begin{aligned} &\sum_{m \in \mathfrak{M}_l} \int_{B_{\tau_m}(\zeta_m)} \left( \sum_{k \in \mathbb{L}_m} |\tilde{\mathbf{a}}_k(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_l - \xi_k)) \right)^{1/r'} dz = \sum_{m \in \mathfrak{M}_l} \int_{B_{\tau_m}(\zeta_m)} \tilde{\mathbf{a}}_l(z) dz \\ &= \sum_{m \in \mathfrak{M}_l} \int_{B_{\tau_m}(\zeta_m)} \left( \sum_{j \in \mathbb{Z}} |\mathbf{a}_j(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_l - \mathbf{c}_j(z))) \right)^{1/r'} dz. \end{aligned}$$

This allows us to conclude that

$$\tilde{\mathbb{M}}_{2R}^+(x_l, \xi_l, s_l) \gtrsim \int_{B_{R s_l}(x_l)} \left( \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_{\Theta^+}(s_l(\xi_l - \mathbf{c}_k(z))) \right)^{1/r'} dz = \mathbb{M}_R^+(x_l, \xi_l, s_l).$$

□

We now have all the tools to prove (2.60) for  $\mathbb{M}_R^+$ . We proceed by interpolation, as described in 2.2.1, between the four (weak) endpoints

$$(p', q') \in \{(\infty, \infty), (\infty, r'), (1, \infty), (1, r')\}.$$

*Proof of bounds (2.60) for  $\mathbb{M}_R^+$ .*

The bound for  $(p', q') = (\infty, \infty)$  follows directly from (2.65) with  $p' = \infty$ .

The bound for  $(p', q') = (\infty, r')$  follows from the locality property 2.21. We must show that for any strip  $D(x, s)$  one has

$$\|\mathbb{M}_R^+ \mathbb{1}_{D(x,s)}\|_{L^{r', \infty}(S^\infty)}^{r'} \lesssim_R \nu(D(x, s)) \|\mathbf{a}\|_{L^\infty(l^{r'})}^{r'}$$

but due to locality and (2.66) we have that

$$\|\mathbb{M}_R^+ \mathbb{1}_{D(x,s)}\|_{L^{r', \infty}(S^\infty)}^{r'} \lesssim_R \|\mathbf{a} \mathbb{1}_{B_{2sR}(x)}\|_{L^{r'}(l^{r'})}^{r'} \leq_R s \|\mathbf{a}\|_{L^\infty(l^{r'})}^{r'}$$

as required.

The bound for  $(p', q') = (1, \infty)$  makes use of the Mass Projection Lemma 2.22. We need to show that for every  $\omega > 0$  there exists  $K_\omega \subset \mathbb{X}$  such that

$$\nu(K_\omega) \lesssim_R \omega^{-1} \|\mathbf{a}\|_{L^1(l^{r'})} \quad \|\mathbb{M}_R^+ \mathbb{1}_{\mathbb{X} \setminus K_\omega} \mathbb{1}_{D(x,s)}\|_{L^\infty(S^\infty)} \lesssim \omega.$$

for any strip  $D(x, s)$ . Let  $\mathcal{K}_\omega = \{z \in \mathbb{R}: M(\|\mathbf{a}\|_{l^{r'}})(z) > \omega\}$  where  $M$  is the Hardy-Littlewood Maximal function (2.25). The set  $\mathcal{K}_\tau$  is open and in particular is a finite union of intervals  $\mathcal{K}_\omega = \bigcup_{m=1}^M B_{\tau_m}(\zeta_m)$ . Let

$$K_\omega := \bigcup_{m=1}^M D(\zeta_m, 9\tau_m) \implies \nu(K_\omega) \lesssim \sum_{m=1}^M 2\tau_m = |\mathcal{K}_\omega| \lesssim \omega^{-1} \|\mathbf{a}\|_{L^1(l^{r'})}$$

by the weak  $L^1$  bound on the Hardy-Littlewood maximal function.

For any tent  $T(y, \eta, t) \not\subset D(\zeta_m, 3\tau_m)$  apply Lemma 2.22 with respect to the the strips  $(D(\zeta_m, 3\tau_m))_{m \in \{1, \dots, M\}}$  and the one tent  $T(\xi, x, s)$ . By construction we obtain a function  $\tilde{\mathbf{a}}$  such that  $\|\tilde{\mathbf{a}}\|_{L^\infty(l^{r'})} \lesssim \omega$ . Using the statement of the Lemma and bound (2.65) we have

$$\mathbb{M}_R^+(y, \eta, t) \leq \tilde{\mathbb{M}}_{2R}^+(y, \eta, t) \lesssim_R \|\tilde{\mathbf{a}}\|_{L^\infty(l^{r'})} \lesssim \omega$$

as required.

The proof of the case  $(p', q') = (1, r')$  goes along the same lines. Let us suppose, without loss of generality, that  $\mathbf{a} \in C_c^\infty(l^{r'})$ . We need to show that for every  $\omega > 0$  there exists  $K_\omega \subset \mathbb{X}$  such that

$$\nu(K_\omega) \lesssim_R \omega^{-1} \|\mathbf{a}\|_{L^1(l^{r'})} \quad \|\mathbb{M}_R^+ \mathbb{1}_{\mathbb{X} \setminus K_\omega} \mathbb{1}_{D(x,s)}\|_{L^{r', \infty}(S^\infty)}^{r'} \lesssim \nu(D(x, s)) \omega^{r'}.$$

Choose  $K_\omega = \bigcup_{m=1}^M D(\zeta_m, 9\tau_m)$  as before. Let  $\lambda > 0$  and set

$$\bar{E}_{\lambda, R} = E_{\lambda, R} \cap (D(x, s) \setminus K_\omega) \quad E_{\lambda, R} = \{(y, \eta, t): \mathbb{M}_R^+ > \lambda\}.$$

The Covering Lemma 2.19 can be applied to  $\overline{E_{\lambda,R}}$  with  $Q > 2R$  sufficiently large yielding a covering  $(T(x_l, \xi_l, s_l))_{l \in \{1, \dots, L\}}$  such that  $\bigcup_{l=1}^L 3Q^3 T(x_l, \xi_l, s_l) \supset \overline{E_{\lambda,R}}$  with the tents  $T(x_l, \xi_l, s_l)$  that are pairwise  $Q^+$ -disjoint. Now apply the Mass Projection Lemma 2.22 with respect to the strips  $(D(\zeta_m, 3\sigma_m))_{m \in \{1, \dots, M\}}$  and the tents  $T(x_l, \xi_l, s_l)_{l \in \{1, \dots, L\}}$ . The resulting  $\tilde{\mathfrak{a}}$  satisfies  $\|\tilde{\mathfrak{a}}\|_{L^\infty(I^{r'})} \lesssim \omega$  while

$$\tilde{\mathfrak{M}}_{2R}^+(x_l, \xi_l, s_l) \geq \mathfrak{M}_R^+(x_l, \xi_l, s_l) \geq \lambda.$$

Using the bound (2.63) and the locality property 2.21 we have that

$$\mu(\overline{E_{\lambda,R}}) \lesssim_R \sum_{l=1}^L s_l \lesssim \lambda^{-r'} \|\tilde{\mathfrak{a}}\|_{L^{r'}(I^{r'})}^{r'} \lesssim s \omega^{r'} \lambda^{-r'}.$$

This concludes the proof.  $\square$

## 2.5 Proof of Theorem 2.3

In the previous section the bounds (2.60) were shown to hold for the auxiliary embedding  $\mathfrak{M}$ . To prove Theorem 2.3 it is sufficient to show that the values of  $\mathfrak{M}$  control  $\|\cdot\|_{S_m}$ . More specifically we require the following result.

**Proposition 2.23.** *Given any union of strips  $K$  and a union of tents  $E$  such that*

$$\mathfrak{M}(y, \eta, t) \leq \lambda \quad \forall (y, \eta, t) \in \mathbb{X} \setminus (K \cup E) \quad (2.70)$$

*the bound  $\|\mathbb{A} \mathbb{1}_{\mathbb{X} \setminus (K \cup E)}\|_{S_m} \lesssim \lambda$  holds.*

Assuming that the above statement holds, Theorem 2.3 follows by the monotonicity property of outer  $L^p$  sizes 2.11.

The above proposition follows from showing that the required bound holds for all local sizes:  $\|\mathbb{A} \mathbb{1}_{\mathbb{X} \setminus (K \cup E)}\|_{S_m(T)} \lesssim \lambda$ . The proof is divided into two parts relative to showing  $L^1$ -type bounds over  $T^{(i)}$  and  $L^{(2)}$  type bounds over  $T^{(e)}$  (see (2.42)).

The former part uses crucial disjointness properties related to the conditions (2.17) on the truncated wave packets.

The latter part depends on the fact that the sizes over a single tent  $T$  resembles an  $L^2$  estimate for variational truncation of the Hilbert transform or of a square function in the spirit of [JSW08]. We will elaborate on this variational estimate in Lemma 2.24 in the following part on technical preliminaries.

The proof also involves a crucial stopping time argument. Similarly to the rest of the paper we avoid discretization and formulate a continuous version of this argument that we isolate Lemma 2.27 below.

### 2.5.1 Technical preliminaries

The following variational truncation bounds are a slightly modified version of the results appearing in [JSW08].

**Lemma 2.24** (Variational truncations of singular integral operators [JSW08]). *For any function  $H \in L^p(\mathbb{R})$  and  $\sigma \in [0, \infty)$  let us define the variational truncation operator*

$$\mathcal{V}_\sigma^r H(z) = \sup_{\sigma < t_1 < \dots < t_k < \dots} \left( \sum_k |H * \Upsilon_{t_{k+1}}(z) - H * \Upsilon_{t_k}(z)|^r \right)^{1/r} \quad (2.71)$$

where

$$\Upsilon \in S(\mathbb{R}), \int_{\mathbb{R}} \Upsilon(z) dz = 1 \text{ and } \Upsilon_t(z) := t^{-1} \Upsilon\left(\frac{z}{t}\right).$$

If  $r > 2$  and for any  $p \in (1, \infty)$ , above operator satisfies the bounds

$$\|\mathcal{V}_\sigma^r H\|_{L^p} \lesssim_{r,p} \|H\|_{L^p} \quad (2.72)$$

and if  $\sigma > 0$  then

$$\mathcal{V}_\sigma^r H(z) \lesssim_{r,p} \int_{B_\sigma(z)} M(\mathcal{V}_\sigma^r H)(z') dz' \quad (2.73)$$

where  $M$  is the Hardy-Littlewood maximal function. The implicit constants are allowed to depend on  $\Upsilon$ .

We record some useful properties of so called *convex regions* of tents.

**Definition 2.25** (Convex regions). *A convex region of a tent is a subset  $\Omega \subset T(x, \xi, s)$  of a tent of the form*

$$\Omega := \bigcup_{\theta \in \Theta} \Omega_\theta := \{(y, \xi + \theta t^{-1}, t) \in T(x, \xi, s) : t > \sigma_\theta(y)\}. \quad (2.74)$$

for some function  $\sigma_\theta(y) : \Theta \times B_s(x) \rightarrow [0, s]$ .

Given any tent  $T \in \mathbb{T}$ , any collection of strips  $\mathcal{D}$ , and any collection of tents  $\mathcal{T}$ , the set

$$\Omega = T \setminus \left( \bigcup_{D \in \mathcal{D}} D \cup \bigcup_{T \in \mathcal{T}} T \right) \subset T$$

is a convex region of the tent  $T$ . With the next lemma we show that the bound (2.70) on a convex regions can be extended to larger regions with scale bound  $\sigma$  that is Lipschitz in the space variable.

**Lemma 2.26** (Lipschitz convex regions). *Let  $T(x, \xi, s) \in \mathbb{T}$  be a tent and  $\Omega = \bigcup_{\theta \in \Theta} \Omega_\theta \subset T(x, \xi, s)$  be a convex region as defined in (2.74) and let us fix a constant  $L > 2$ . For every  $\theta \in \Theta$  such that the bound*

$$\begin{aligned} \mathbb{M}(y, \eta, t) &\leq \lambda \quad \forall (y, \eta, t) \in \Omega_\theta \\ \text{holds for } \Omega_\theta &= \{(y, \xi + \theta t^{-1}, t) \in T(x, \xi, s) : t > \sigma_\theta(y)\} \neq \emptyset \end{aligned}$$

there exists a Lipschitz function  $\tilde{\sigma}_\theta : \mathbb{R} \rightarrow \mathbb{R}^+$  with Lipschitz constant  $L^{-1} < 1/2$  such that

$$\min(2s; L^{-1} \text{dist}(y; B_s(x))) \leq \tilde{\sigma}_\theta(y) \leq 2s \quad \forall y \in \mathbb{R} \quad (2.75)$$

$$\tilde{\sigma}_\theta(y) \leq \sigma_\theta(y) \quad \forall y \in B_s(x) \quad (2.76)$$

and

$$sW_s(x - y)\mathbb{M}(y, \xi + \theta t^{-1}, t) \lesssim_L \lambda \quad \forall y \in \mathbb{R}, t \in (\tilde{\sigma}_\theta(y), 3s). \quad (2.77)$$

*Proof.* Fix  $\theta \in \Theta$  such that  $\Omega_\theta$  is non-empty and let us drop the dependence on  $\theta$  from the notation by simply writing  $\sigma(y)$  in place of  $\sigma_\theta(y)$ . Let us set

$$\tilde{\sigma}(y) := \min(2s; \tilde{\sigma}(y)) \quad \text{with } \tilde{\sigma}(y) := \inf_{y' \in B_s(x)} \max\left(\sigma(y'); \frac{|y - y'|}{L}\right) \quad (2.78)$$

Clearly, this defines a function on  $\mathbb{R}$  such that conditions (2.75) and (2.76) hold. The defined function is  $L^{-1}$ -Lipschitz. It is sufficient to show that  $\tilde{\sigma}$  is  $L^{-1}$ -Lipschitz: for any  $y \in \mathbb{R}$  and  $\varepsilon > 0$  there exists  $y' \in B_s(x)$  such that

$$\tilde{\sigma}(y) \geq (1 + \varepsilon)^{-1} \max\left(\sigma(y'); \frac{|y - y'|}{L}\right)$$

and thus for any  $y'' \in \mathbb{R}$  one has

$$\begin{aligned} \tilde{\sigma}(y'') &\leq \max\left(\sigma(y'); \frac{|y'' - y'|}{L}\right) \leq \max\left(\sigma(y'); \frac{|y - y'|}{L}\right) + \frac{|y'' - y|}{L} \\ &\leq (1 + \varepsilon)\tilde{\sigma}(y) + \frac{|y'' - y|}{L}. \end{aligned}$$

Since  $\varepsilon > 0$  was arbitrary and one can invert the role of  $y''$  and  $y$  in the above reasoning we obtain that  $|\tilde{\sigma}(y'') - \tilde{\sigma}(y)| \leq \frac{|y'' - y|}{L}$  as required.

Let us now check that (2.77) holds. Suppose that  $y \in \mathbb{R}$  and  $t \in (\tilde{\sigma}(y), 3s]$ . Let us distinguish the cases  $t \in (\tilde{\sigma}(y), 2s)$  and  $t \in [2s, 3s]$ . In the first case there exists  $y' \in B_s(x)$  and  $t' \in (\sigma_\theta(y'), s)$  such that  $t' \in (t/2, t)$  and  $|y - y'| < 2Lt$  and thus it follows that  $|x - y| < 2Ls$ . It follows that

$$\begin{aligned} W_t(z - y) &\lesssim_L W_t(z - y') \lesssim W_{t'}(z - y') & \forall z \in \mathbb{R} \\ \text{thus } sW_s(x - y)W_t(z - y) &\lesssim_L W_{t'}(z - y') & \forall z \in \mathbb{R} \end{aligned}$$

In the case that  $t \in [2s, 3s]$  there also exists  $y' \in B_s(x)$  and  $t' \in (\sigma_\theta(y'), s)$  such that  $t' \in (t/2, t)$  since  $\Omega_\theta \neq \emptyset$ . It follows from (2.78) that  $|y' - y| > 2Ls$  so  $|x - y| \approx_L |y' - y|$  so for all  $z \in \mathbb{R}$

$$sW_s(x - y)W_t(z - y) \lesssim_L sW_s(y' - y)W_s(z - y) \lesssim W_{t'}(z - y').$$

Thus, since in both cases  $(y', \xi + \theta t'^{-1}, t') \in \Omega$  we have by the definition (2.59) of  $\mathbb{M}$  that

$$sW_s(x - y)\mathbb{M}(y, \xi + \theta t^{-1}, t) \lesssim_L \mathbb{M}(y', \xi + \theta t'^{-1}, t) \leq \lambda$$

as required.  $\square$

The next technical lemma will be used as a continuous stopping time argument. It relates the Lipschitz assumption on enlarged convex regions of the previous statement with a crucial measurability estimate.

**Lemma 2.27** (Continuous stopping time). *Let  $\sigma : \mathbb{R} \rightarrow \mathbb{R}^+$  be a Lipschitz function with Lipschitz constant  $L^{-1} < 1$ . Then the function*

$$\rho_\sigma(z) := \int_{\mathbb{R}} \frac{1}{2\sigma(x)} \mathbb{1}_{B_{\sigma(x)}}(z - x) dx$$

satisfies  $\left(1 + \frac{2}{L-1}\right)^{-1} < \rho_\sigma(z) < 1 + \frac{2}{L-1}$  and in particular for any non-negative function  $h(z)$  the bounds

$$\int_{\mathbb{R}} h(z) dz \approx_L \int_{\mathbb{R}} \int_{B_{\sigma(x)}(x)} h(z) dz dx.$$

hold.

*Proof.* Since  $\sigma$  is  $L^{-1}$ -Lipschitz, for any  $z \in \mathbb{R}$  we have that

$$B_{(1+L^{-1})^{-1}\sigma(z)}(z) \subseteq \{x: z \in B_{\sigma(x)}(x)\} \subseteq B_{(1-L^{-1})^{-1}\sigma(z)}(z).$$

By the same reason on  $\{x: z \in B_{\sigma(x)}(x)\}$  we have that

$$(1+L^{-1})^{-1}\sigma(z) \leq \sigma(x) \leq (1-L^{-1})^{-1}\sigma(z).$$

The conclusion follows.  $\square$

## 2.5.2 Proof of Proposition 2.23

Let  $T = T(x, \xi, s)$  be a tent and suppose that  $K$  and  $E$  are as in 2.23. Since the statement of Proposition 2.23 is invariant under time and frequency translations, we may assume, without loss of generality, that  $T$  is centered at the origin i.e.  $T = T(0, 0, s)$ . If  $T \setminus (K \cup E) = \emptyset$  there is nothing to prove. Let us set

$$\begin{aligned} \Theta_* &= \{\theta \in \Theta: \exists(y, \theta t^{-1}, t) \in T(0, 0, s) \setminus (K \cup E)\}, \\ \Theta_*^{(i)} &:= \Theta^{(i)} \cap \Theta_* \quad \Theta_*^{(e)} := \Theta^{(e)} \cap \Theta_*. \end{aligned}$$

For  $\theta \in \Theta_*$ , using Lemma 2.26 we may assume that there exists a  $L^{-1}$ -Lipschitz function  $\sigma_\theta: \mathbb{R} \rightarrow (0, 2s]$ , with  $L > 4$  sufficiently large to be chosen later, that satisfies condition (2.75) such that

$$sW_s(y)\mathbb{M}(y, \theta t^{-1}, t) \lesssim \lambda \quad \forall y \in \mathbb{R}, \theta \in \Theta, t \in (\sigma(y), 3s). \quad (2.79)$$

Let us set  $\Omega = \bigcup_{\theta \in \Theta} \Omega_\theta$ ,  $\Omega^{(i)} = \bigcup_{\theta \in \Theta^{(i)}} \Omega_\theta$ , and  $\Omega^{(e)} = \bigcup_{\theta \in \Theta^{(e)}} \Omega_\theta$  with

$$\Omega_\theta = \begin{cases} \{(y, \theta t^{-1}, t) \in T(0, 0, s): t > \sigma_\theta(y)\} & \theta \in \Theta_* \\ \emptyset & \text{otherwise.} \end{cases}$$

We need to show that

$$\|\mathbb{A} \mathbb{1}_{\mathbb{X} \setminus (K \cup E)}\|_{S_m(T)} \leq \|\mathbb{A} \mathbb{1}_\Omega\|_{S_m(T)} \lesssim \lambda \quad \forall T \in \mathbb{T}$$

or equivalently (see (2.42)) that

$$\|\mathbb{A} \mathbb{1}_{\Omega^{(i)}}\|_{S^1(T^{(i)})} \lesssim \lambda \quad \|\mathbb{A} \mathbb{1}_\Omega\|_{S^2(T)} \lesssim \lambda.$$

In this proof all our implicit constants depend on the choice of  $L$ .

Let us fix a choice of left truncated wave packets  $\Psi_{y, \eta, t}^{c_-, c_+}(z)$  in the defining expression (2.14). We will show that the statement holds in this case. The proof for right truncated wave packets is symmetric.

Comparing the definitions (2.14) and (2.59) for  $\mathbb{A}$  and  $\mathbb{M}$  respectively, it follows from the bound  $|\Psi_{y, \eta, t}^{c_k(z), c_{k+1}(z)}(z)| \leq W_t(z - y)$  that

$$\mathbb{A}(y, \eta, t) \lesssim \mathbb{M}(y, \eta, t), \quad \|\mathbb{A} \mathbb{1}_\Omega\|_{S^\infty(T)} := \sup_{(y, \eta, t) \in \Omega} \mathbb{A}(y, \eta, t) \lesssim \lambda. \quad (2.80)$$

This implies

$$\frac{1}{s} \iiint_{\substack{(y, \eta, t) \in \Omega \\ \eta < 0}} |\mathbb{A}(y, \eta, t)|^2 \lesssim \lambda \frac{1}{s} \iiint_{\substack{(y, \eta, t) \in \Omega \\ \eta < 0}} |\mathbb{A}(y, \eta, t)| dy d\eta dt \quad (2.81)$$

and thus we may assume that  $\alpha^- = \beta^- < 0 < \beta^+ < \alpha^+$  and we can reduce to showing

$$\|\mathbb{A} \mathbb{1}_{\Omega^{(i)}}\|_{S^1(T^{(i)})} \lesssim \lambda \quad \|\mathbb{A} \mathbb{1}_{\Omega^{(e)}}\|_{S^2(T^{(e)})} \lesssim \lambda. \quad (2.82)$$



**Proof of the first inequality of (2.82)**

It holds that

$$\begin{aligned} \|A \mathbb{1}_{\Omega^{(i)}}\|_{S^1(T^{(i)})} &\approx \iiint_{\Omega^{(i)}} \left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbf{a}_k(z) \Psi_{y,\eta,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dz \right| dy d\eta dt \\ &\leq \frac{1}{s} \int_{\theta \in \Theta_*^{(i)}} \int_{y \in B_s} \int_{t=\sigma_\theta(y)}^s \int_{z \in \mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \left| \Psi_{y,\theta t^{-1},t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) \right| dz \frac{dt}{t} dy d\theta. \end{aligned}$$

According to (2.17) The wave-packet  $\Psi_{y,\theta t^{-1},t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z)$  vanishes unless  $\theta - t\mathbf{c}_k(z) \in B_\varepsilon(d)$  and  $t\mathbf{c}_{k+1}(z) - \theta > d'$ . Since  $\theta \in \Theta^{(i)} \subset [-d', d - \varepsilon]$ , the integrand vanishes unless  $\mathbf{c}_k(z) < 0 < \mathbf{c}_{k+1}(z)$ . Let  $k_z^* \in \mathbb{Z}$  be the index, if it exists, such that this inequality holds and set  $a^*(z) := \mathbf{a}_{k_z^*}(z)$ ,  $c^*(z) = \mathbf{c}_{k_z^*}(z)$ . If no such index exists simply set  $a^*(z) = 0$ .

Using that given  $t < s$  and  $y \in B_s$  one has

$$\left| \Psi_{y,\theta t^{-1},t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) \right| \lesssim s W_s(z) t W_t(z - y)^2 \leq W_t(z - y)$$

and using the statement of Lemma 2.27 we have that

$$\begin{aligned} \|A \mathbb{1}_{\Omega^{(i)}}\|_{S^1(T^{(i)})} &\lesssim \frac{1}{s} \int_{\theta \in \Theta_*^{(i)}} \int_{y \in B_s} \int_{t=\sigma_\theta(y)}^s \int_{x \in \mathbb{R}} \int_{z \in B_{\sigma_\theta(x)}(x)} |a^*(z)| s W_s(z) t W_t(y - z)^2 \\ &\quad \times \mathbb{1}_{B_\varepsilon(d)}(\theta - tc^*(z)) dz dx \frac{dt}{t} dy d\theta = I + II \end{aligned}$$

where

$$\begin{aligned} I &:= \frac{1}{s} \int_{\theta \in \Theta_*^{(i)}} \int_{x \in \mathbb{R}} \int_{z \in B_{\sigma_\theta(x)}(x)} |a^*(z)| \int_{t=(1-2/L)\sigma_\theta(x)}^s \int_{y \in B_s} W_t(y - z) dy \\ &\quad \times \mathbb{1}_{B_\varepsilon(d)}(\theta - tc^*(z)) \frac{dt}{t} dz dx d\theta \\ II &:= \frac{1}{s} \int_{\theta \in \Theta_*^{(i)}} \int_{x \in \mathbb{R}} \int_{y \in B_s} \int_{t=\sigma_\theta(y)}^{(1-2/L)\sigma_\theta(x)} \int_{z \in B_{\sigma_\theta(x)}(x)} |a^*(z)| s W_s(z) t W_t(y - z)^2 \\ &\quad \times \mathbb{1}_{B_\varepsilon(d)}(\theta - tc^*(z)) dz \frac{dt}{t} dy dx d\theta \end{aligned}$$

Suppose that  $L > 2 \frac{\alpha^+ - \alpha^-}{\alpha^+ - d - \varepsilon}$  so that for any  $c \in \mathbb{R}$  one has

$$\begin{cases} \theta - tc \in B_\varepsilon(d) \\ t > (1 - 2/L)\sigma_\theta(x) \end{cases} \implies \theta - \sigma_\theta(x)c \in \Theta. \quad (2.83)$$

We begin by estimating the term  $I$ . Notice that if  $|x| > 2Ls$  then integrand vanishes. We bound  $I$  by the auxiliary embedding map (2.59) as follows:

$$\begin{aligned} I &\lesssim \frac{1}{s} \int_{\theta \in \Theta_*^{(i)}} \int_{x \in \mathbb{R}} \int_{z \in B_{\sigma_\theta(x)}(x)} |a^*(z)| \int_{t=(1-2/L)\sigma_\theta(x)}^s \mathbb{1}_{B_\varepsilon(d)}(\theta - tc^*(z)) \frac{dt}{t} dz dx d\theta \\ &\lesssim \frac{1}{s} \int_{\theta \in \Theta_*^{(i)}} \int_{x \in B_{2Ls}} \int_{B_{\sigma_\theta(x)}(x)} |a^*(z)| \mathbb{1}_{\Theta_*}(\theta - \sigma_\theta(x)\mathbf{c}_k(z)) \ln \left( \frac{\beta^+ - \theta + 3\varepsilon}{\beta^+ - \theta} \right) dz dx d\theta \end{aligned}$$

$$\leq \frac{1}{s} \int_{\theta \in \Theta_*^{(i)}} \int_{x \in B_{2Ls}} \mathbb{M}(x, \theta \sigma_\theta(x)^{-1}, \sigma_\theta(x)) dx \ln \left( \frac{\beta^+ - \theta + 3\varepsilon}{\beta^+ - \theta} \right) d\theta \lesssim \lambda.$$

The last inequality holds since  $(x, \theta \sigma_\theta(x)^{-1}, \sigma_\theta(x)) \in \Omega_\theta$  and follows from (2.79). We now estimate the term  $II$ . Notice that

$$(1 - 2/L)\sigma_\theta(x) > \sigma_\theta(y) \implies |x - y| \geq L(\sigma_\theta(x) - \sigma_\theta(y)) > 2\sigma_\theta(x) \quad (2.84)$$

Thus if  $z \in B_{\sigma_\theta(x)}(x)$  then  $|y - z| > \sigma_\theta(x) > t$ ,  $|x - y| \approx |y - z|$ , and also  $sW_s(z) \lesssim sW_s(x)$  so

$$\begin{aligned} II &\lesssim \int_{\theta \in \Theta_*^{(i)}} \int_{x \in \mathbb{R}} W_s(x) \int_{y \in B_s} \int_{t=\sigma_\theta(y)}^{(1-2/L)\sigma_\theta(x)} t W_t(y-x) \\ &\quad \times \int_{B_{\sigma_\theta(x)}(x)} |a^*(z)| W_t(y-z) \mathbb{1}_{B_\varepsilon(d)}(\theta - tc^*(z)) dz \frac{dt}{t} dy dx d\theta \\ &\lesssim \int_{\theta \in \Theta_*^{(i)}} \int_{x \in \mathbb{R}} W_s(x) \int_{y \in B_s} \int_{t=\sigma_\theta(y)}^{(1-2/L)\sigma_\theta(x)} \frac{t}{2\sigma_\theta(x)} W_t(y-x) \mathbb{M}(y, \theta t^{-1}, t) \frac{dt}{t} dy dx d\theta. \end{aligned}$$

Since the inmost integral vanishes unless  $|y - x| > 2\sigma_\theta(x)$ , we have that

$$\int_{t=\sigma_\theta(y)}^{(1-2/L)\sigma_\theta(x)} \frac{t}{2\sigma_\theta(x)} W_t(y-x) \frac{dt}{t} \lesssim W_{\sigma_\theta(x)}(y-x)$$

and so using (2.79) we obtain

$$\begin{aligned} II &\lesssim \lambda \int_{\theta \in \Theta_*^{(i)}} \int_{x \in \mathbb{R}} W_s(x) \int_{y \in B_s} W_{\sigma_\theta(x)}(y-x) dy dx d\theta \\ &\lesssim \lambda \int_{\theta \in \Theta_*^{(i)}} \int_{x \in \mathbb{R}} W_s(x) dx d\theta \lesssim \lambda. \end{aligned}$$

This concludes the proof for the first bound of (2.82).

### Proof of the second inequality of (2.82)

As noted in (2.81) we may suppose that  $\Theta^{(e)} = [\beta^+, \alpha^+)$ ; for ease of notation set  $\Omega^{(e)} = \Omega \cap T^{(e)}$  so the required quantity to bound becomes  $\|\mathbb{A} \mathbb{1}_{\Omega^{(e)}}\|_{S^2(T^{(e)})}$ . We concentrate on showing the dual bound

$$\frac{1}{s} \left| \iiint_{\mathcal{X}} h(y, \eta, t) \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbf{a}_k(z) \Psi_{y, \eta, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dz dy d\eta dt \right| \lesssim \lambda \frac{\|h(y, \eta, t)\|_{L^2}}{s^{1/2}} \quad (2.85)$$

for any  $h \in C_c^\infty(\Omega^{(e)})$  where the  $\|\cdot\|_{L^2}$  is the classical Lebesgue  $L^2$  norm relative to the measure  $dy d\eta dt$ . A change of variables and the Minkowski inequality give

$$\begin{aligned} &\frac{1}{s} \left| \iiint_{\mathcal{X}} h(y, \eta, t) \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbf{a}_k(z) \Psi_{y, \eta, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dz dy d\eta dt \right| \\ &\leq \frac{1}{s} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \left| \iiint_{\Omega^{(e)}} h(y, \eta, t) \Psi_{y, \eta, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy d\eta dt \right| dz \\ &\leq \frac{1}{s} \int_{\theta \in \Theta_*^{(e)}} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \left| \int_{y \in B_s} \int_{t=\sigma_\theta(y)}^s h(y, t^{-1}\theta, t) \Psi_{y, \theta t^{-1}, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy \frac{dt}{t} \right| dz d\theta. \end{aligned}$$

On the other hand the Hölder inequality gives that

$$\begin{aligned} \int_{\theta \in \Theta_*^{(e)}} \|h(y, t^{-1}\theta, t)\|_{L^2_{(dy dt/t)}} d\theta &= \int_{\theta \in \Theta_*^{(e)}} \left( \int_{y \in B_s} \int_{t=\sigma_\theta(y)}^s |h(y, t^{-1}\theta, t)|^2 dy \frac{dt}{t} \right)^{\frac{1}{2}} d\theta \\ &\lesssim \|h(y, \eta, t)\|_{L^2} \end{aligned}$$

where  $\|\cdot\|_{L^2_{(dy dt/t)}}$  is the classic Lebesgue  $L^2$  norm with respect to the measure  $dy \frac{dt}{t}$ . Thus (2.85) follows by showing

$$\begin{aligned} \frac{1}{s} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \left| \int_{y \in B_s} \int_{t=\sigma_\theta(y)}^s h(y, t^{-1}\theta, t) \Psi_{y, t^{-1}\theta, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy \frac{dt}{t} \right| dz & \quad (2.86) \\ &\lesssim \lambda \frac{\|h(y, t^{-1}\theta, t)\|_{L^2_{(dy dt/t)}}}{s^{1/2}} \end{aligned}$$

with a constant uniform in  $\theta \in \Theta_*^{(e)}$ . For sake of notation from now on we will omit the dependence on  $\theta$  by writing

$$h(y, t) := h(y, t^{-1}\theta, t) \quad \Psi_{y, t}^{c_-, c_+}(z) := \Psi_{y, t^{-1}\theta, t}^{c_-, c_+}(z) \quad \sigma(x) := \sigma_\theta(x).$$

Using the above notation and Lemma 2.27 we write

$$\frac{1}{s} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \left| \int_{y \in B_s} \int_{t=\sigma(y)}^s h(y, t) \Psi_{y, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy \frac{dt}{t} \right| dz \lesssim I + II$$

where

$$\begin{aligned} I &:= \frac{1}{s} \int_{x \in B_{2Ls}} \int_{z \in B_{\sigma(x)}(x)} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \left| \int_{y \in B_s} \int_{t=(1-2/L)\sigma(x)}^s h(y, t) \Psi_{y, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy \frac{dt}{t} \right| dz dx \\ II &:= \frac{1}{s} \int_{x \in \mathbb{R}} \int_{z \in B_{\sigma(x)}(x)} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \left| \int_{y \in B_s} \int_{t=\sigma(y)}^{(1-2/L)\sigma(x)} h(y, t) \Psi_{y, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy \frac{dt}{t} \right| dz dx \end{aligned}$$

We start with bounding  $I$ . Suppose that  $L > 1$  is chosen large enough so that (2.83) holds and recall that  $\Psi_{y, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) = 0$  unless  $\theta - t\mathbf{c}_k(z) \in B_\varepsilon(d)$ . We thus have

$$\begin{aligned} I &= \frac{1}{s} \int_{x \in B_{2Ls}} \int_{z \in B_{\sigma(x)}(x)} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \left| \int_{y \in B_s} \int_{t=(1-\frac{2}{L})\sigma(x)}^s h(y, t) \Psi_{y, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy \frac{dt}{t} \right| dz dx \\ &\leq \frac{1}{s} \int_{x \in B_{2Ls}} \int_{B_{\sigma(x)}(x)} \left( \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_\Theta(\theta - \sigma(x)\mathbf{c}_k(z)) \right)^{1/r'} \mathcal{H}_x(z) dz dx \\ &\leq \frac{1}{s} \int_{x \in B_{2Ls}} \mathfrak{M}(x, \theta\sigma(x)^{-1}, \sigma(x)) \sup_{z \in B_{\sigma(x)}(x)} \mathcal{H}_x(z) dx \leq \frac{\lambda}{s} \int_{x \in B_{2Ls}} \sup_{z \in B_{\sigma(x)}(x)} \mathcal{H}_x(z) dx, \end{aligned}$$

where

$$\mathcal{H}_x(z) := \left( \sum_{k \in \mathbb{Z}} \left| \int_{y \in B_s} \int_{t=(1-\frac{2}{L})\sigma(x)}^s h(y, t) \Psi_{y, t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy \frac{dt}{t} \right|^r \right)^{1/r}.$$

We claim that

$$\begin{aligned} \mathcal{H}_x(z) &\lesssim \mathcal{V}_{\sigma(x)}^r H_s(z) + \mathcal{E}_{\sigma(x)}(z) & H_\tau(z) &:= \int_{t=0}^\tau \int_{y \in B_s} h(y, t) \Psi_{y,t}^{0,+\infty}(z) dy \frac{dt}{t} \\ \mathcal{E}_{\sigma(x)}(z) &:= \left( \int_{(1-2/L)\sigma(x)}^s |h^*(z, t)|^2 \frac{dt}{t} \right)^{1/2} & h^*(z, t) &:= \int_{\mathbb{R}} |h(y, t)| W_t(z - y) dy \end{aligned} \quad (2.87)$$

with  $\mathcal{V}_{\sigma(x)}^r$  defined in Lemma 2.24, and that

$$\|\mathcal{E}_0\|_{L^2} \lesssim \|h\|_{L^2_{(dydt/t)}} \quad \sup_{z \in B_{\sigma(x)}(x)} \mathcal{E}_{\sigma(x)}(z) \lesssim \int_{B_{2\sigma(x)}(z)} \mathcal{E}_0(z) dz \quad (2.88)$$

$$\|H_s\|_{L^2} \lesssim \|h\|_{L^2_{(dydt/t)}} \quad \sup_{z \in B_{\sigma(x)}(x)} \mathcal{H}_x(z) dx \lesssim \int_{B_{2\sigma(x)}(x)} M\mathcal{V}^r H_s(z) dz. \quad (2.89)$$

This would provide us with the required bounds for  $I$ . As a matter of fact, according to Lemma 2.27 and 2.24 we have that

$$\begin{aligned} I &\lesssim \frac{\lambda}{s} \int_{x \in B_{2Ls}} \int_{B_{2\sigma(x)}(x)} \left( M\mathcal{V}^r H_s(z) + \mathcal{E}_{\sigma(x)}(z) \right) dz \lesssim_R \frac{\lambda}{s^{1/2}} \left( \|M\mathcal{V}^r H_s\|_{L^2} + \|\mathcal{E}_0(z)\|_{L^2} \right) \\ &\lesssim \frac{\lambda}{s^{1/2}} \left( \|H_s(z)\|_{L^2} + \|\mathcal{E}_0(z)\|_{L^2} \right) \lesssim \lambda \frac{\|h\|_{L^2_{(dydt/t)}}}{s^{1/2}} \end{aligned}$$

as required.

The first bound of (2.88) follows by the Young inequality and Fubini:

$$\begin{aligned} \|\mathcal{E}_0\|_{L^2} &\leq \iint_{\mathbb{R} \times \mathbb{R}^+} \left| \int_{\mathbb{R}} |h(y, t)| W_t(z - y) dy \right|^2 dz \frac{dt}{t} \\ &\leq \int_{\mathbb{R}^+} \int_{\mathbb{R}} |h(y, t)|^2 dy \left( \int_{\mathbb{R}} W_t(z - y) dy \right)^2 \frac{dt}{t} \lesssim \|h\|_{L^2_{(dydt/t)}}^2. \end{aligned}$$

The second bound follows from the fact that for small enough  $\varepsilon > 0$  and as long as  $|z - z'| < \varepsilon t$  the bound

$$\begin{aligned} |h^*(z, t) - h^*(z', t)| &\leq \int_{\mathbb{R}} |h(y, t)| |W_t(z - y) - W_t(z' - y)| dy \\ &\leq 2^{-100} \int_{\mathbb{R}} |h(y, t)| W_t(z - y) dy = 2^{-100} h^*(z, t) \end{aligned}$$

holds so similarly

$$\left| \mathcal{E}_{\sigma(x)}(z) - \mathcal{E}_{\sigma(x)}(z') \right| \leq 2^{-100} \mathcal{E}_{\sigma(x)}(z)$$

as long as  $|z - z'| < \varepsilon \sigma(x)$  for some sufficiently small  $\varepsilon > 0$ .

$$\int_{B_{\sigma(x)}(z)} \mathcal{E}_0(z') dz' \gtrsim \int_{B_{\sigma(x)}(z)} \mathcal{E}_{\sigma(x)}(z') dz' \gtrsim_\varepsilon \int_{B_{\varepsilon\sigma(x)}(z)} \mathcal{E}_{\sigma(x)}(z) dz' = \mathcal{E}_{\sigma(x)}(z)$$

and the claim follows.

The first bound of (2.89) uses standard oscillatory integral techniques: notice that for  $t > t'$  one has

$$\left| \int_{\mathbb{R}} \Psi_{y,t}(z) \overline{\Psi_{y',t'}(z)} dz \right| \lesssim \frac{t'}{t} W_t(y - y')$$

so

$$\begin{aligned} \int |H_s(z)|^2(z) &\lesssim 2 \int_{t=0}^s \int_{t'=0}^t \int_{y \in B_s} \int_{y' \in B_s} |h(y,t)| |h(y',t')| W_t(y - y') dy dy' \frac{t'}{t} \frac{dt}{t} \frac{dt'}{t'} \\ &\lesssim \|h\|_{L^2_{(dy dt/t)}}^2. \end{aligned}$$

The second bound follows directly from Lemma 2.24. It remains to show inequality (2.87). Notice that

$$\mathcal{H}_x(z) = \left( \sum_{k \in \mathbb{Z}} \left| \int_{t_k^-(z)}^{t_k^+(z)} \int_{y \in B_s} h(y,t) \Psi_{y,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) dy \frac{dt}{t} \right|^r \right)^{1/r}$$

where for  $k \in \mathbb{Z}$  we set

$$\begin{aligned} t_k^+(z) &:= \sup \left\{ t \in ((1 - 2/L)\sigma(x), s) : \Psi_{y,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) \neq 0 \right\} \\ t_k^-(z) &:= \inf \left\{ t \in ((1 - 2/L)\sigma(x), s) : \Psi_{y,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) \neq 0 \right\}. \end{aligned} \quad (2.90)$$

We have omitted writing the implicit dependence on  $x \in \mathbb{R}$  and we will simply ignore the indexes  $k \in \mathbb{Z}$  for which the above sets are empty. Notice that the intervals  $[t_k^-(z), t_k^+(z)]$  are disjoint. According to the conditions (2.17) on the geometry of truncated wave packets the following bounds hold:

$$t_k^+(z) \mathbf{c}_k(z) \in \overline{B_\varepsilon(\theta - d)} \quad t_k^-(z) \mathbf{c}_{k+1}(z) \geq \theta + d'. \quad (2.91)$$

Using the smoothness conditions (2.15) on the wave packets and writing a Lagrange remainder term we have that

$$\left| \Psi_{y,t}^{\mathbf{c}_k(z), \mathbf{c}_{k+1}(z)}(z) - \Psi_{y,t}^{0,+\infty}(z) \right| \leq \left( |t \mathbf{c}_k(z)| + \max(d'' - \theta - t \mathbf{c}_{k+1}(z), 0) \right) W_t(y - z) \quad (2.92)$$

so the bound

$$\begin{aligned} \mathcal{H}_x(z) &\leq \mathcal{H}_{x,1}(z) + \mathcal{H}_{x,2}(z) \\ \mathcal{H}_{x,1}(z) &:= \left( \sum_{k \in \mathbb{Z}} \left| \int_{y \in B_s} \int_{t=t_k^-(z)}^{t_k^+(z)} h(y,t) \Psi_{y,t}^{0,+\infty}(z) dy \frac{dt}{t} \right|^r \right)^{1/r} \\ &= \left( \sum_{k \in \mathbb{Z}} |H_{t_k^+(z)}(z) - H_{t_k^-(z)}(z)|^r \right)^{1/r} \\ \mathcal{H}_{x,2}(z) &:= \left( \sum_{k \in \mathbb{Z}} \left| \int_{t_k^-(z)}^{t_k^+(z)} h^*(z,t) \left( |t \mathbf{c}_k(z)| + \max(d'' - \theta - t \mathbf{c}_{k+1}(z), 0) \right) \frac{dt}{t} \right|^r \right)^{1/r} \end{aligned}$$

holds. Notice that

$$\int_{t_k^-(z)}^{t_k^+(z)} t^2 |\mathbf{c}_k(z)|^2 \frac{dt}{t} \leq \frac{|t_k^+(z) \mathbf{c}_k(z)|^2}{2} \leq C_{\alpha^+}$$

$$\int_{t_k^-(z)}^{t_k^+(z)} \max(d'' - \theta - tc_{k+1}(z); 0) \frac{dt}{t} \leq \int_{t_k^-(z)}^{d'' - \theta} \frac{dt}{t} \leq \int_{d'+\theta}^{d'' - \theta} \frac{dt}{t} \lesssim C_{d', d, \alpha^+, \beta^+}$$

for some constant  $C_{\alpha^+}$  and  $C_{d', d, \alpha^+, \beta^+}$ . Since  $r > 2$ , Cauchy-Schwartz gives

$$\mathcal{H}_{x,2}(z) \leq \left( \sum_{k \in \mathbb{Z}} \int_{t_k^-(z)}^{t_k^+(z)} h^*(z, t)^2 \frac{dt}{t} \right)^{\frac{1}{2}} \lesssim \mathcal{E}_{\sigma(x)}(z).$$

This is consistent with (2.87). To estimate  $\mathcal{H}_{x,1}$ : introduce a frequency cutoff  $\Upsilon \in S(\mathbb{R})$  such that

$$\widehat{\Upsilon} \in C_c^\infty(B_{\theta+b}) \quad \widehat{\Upsilon} \geq 0 \quad \widehat{\Upsilon} = 1 \text{ on } B_\theta \quad \Upsilon_\tau(z) := \tau^{-1} \Upsilon\left(\frac{z}{\tau}\right).$$

According to (2.16),  $\widehat{\Psi}_{y,t}^{0,+ \infty}$  is supported on  $B_{t^{-1}b}(t^{-1}\theta)$  so one has the following

$$\begin{aligned} \Psi_{y,t}^{0,+ \infty} * \Upsilon_\tau(z) &= \Psi_{y,t}^{0,+ \infty}(z) && \text{if } \frac{t}{\tau} \geq \frac{\theta+b}{\theta} \\ \Psi_{y,t}^{0,+ \infty} * \Upsilon_\tau(z) &= 0 && \text{if } \frac{t}{\tau} < \frac{\theta-b}{\theta+b} \\ |\Psi_{y,t}^{0,+ \infty} * \Upsilon_\tau(z)| &\lesssim W_t(z-y) && \text{if } \frac{\theta-b}{\theta+b} \leq \frac{t}{\tau} < \frac{\theta+b}{\theta}. \end{aligned}$$

Thus

$$|H_\tau - H_s * \Upsilon_\tau(z)| \lesssim \int_{\tau \frac{\theta-b}{\theta+b}}^{\tau \frac{\theta+b}{\theta}} h^*(z, t) \frac{dt}{t}$$

so

$$\begin{aligned} \mathcal{H}_{x,1}(z) &\lesssim \left( \sum_{k \in \mathbb{Z}} |H_s * \Upsilon_{t_k^+(z)} - H_s * \Upsilon_{t_k^-(z)}|^r \right)^{\frac{1}{r}} + \left( \sum_{k \in \mathbb{Z}} \left| \int_{t_k^-(z) \frac{\theta-b}{\theta+b}}^{t_k^-(z) \frac{\theta+b}{\theta}} h^*(z, t) \frac{dt}{t} \right|^r \right)^{\frac{1}{r}} \\ &+ \left( \sum_{k \in \mathbb{Z}} \left| \int_{t_k^+(z) \frac{\theta-b}{\theta+b}}^{t_k^+(z) \frac{\theta+b}{\theta}} h^*(z, t) \frac{dt}{t} \right|^r \right)^{\frac{1}{r}} \lesssim \mathcal{V}_{\sigma(x)}^r H_s(z) + \mathcal{E}_{\sigma(x)}(z) \end{aligned}$$

thus concluding the proof of (2.87) and the bound on the term  $I$ .

The estimate for the term  $II$  can be done in a manner similar to the term  $II$  in for the  $S^1$  part of the size. Recall (2.84) so that in expression for  $II$  one has that  $z \in B_{\sigma(x)}(x)$ ,  $|y-z| > \sigma(x) > t$ , and  $|x-y| \approx |y-z|$ . We also have that  $y \in B_s$  so

$$\Psi_{y,t}^{c_k(z), c_{k+1}(z)}(z) \lesssim sW_s(x) tW_t(z-y)^2$$

and  $\Psi_{y,t}^{c_k(z), c_{k+1}(z)}(z) = 0$  unless  $tc_k(z) < \theta < tc_{k+1}(z)$ , thus

$$\begin{aligned} II &\lesssim \frac{1}{s} \int_{x \in \mathbb{R}} sW_s(x) \int_{y \in B_s} \int_{t=\sigma(y)}^{(1-2/L)\sigma(x)} tW_t(y-x)h(y, t) \int_{B_{\sigma(x)}(x)} \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)| \\ &\quad \times \mathbb{1}_{\Theta}(\theta - tc_k(z)) W_t(z-y) \mathbb{1}_{(c_k(z), c_{k+1}(z))}(t^{-1}\theta) dz \frac{dt}{t} dy dx \\ &\lesssim \int_{x \in \mathbb{R}} W_s(x) \int_{y \in B_s} \int_{t=\sigma(y)}^{(1x-2/L)\sigma(x)} \frac{t}{2\sigma(x)} W_t(y-x)h(y, t) \end{aligned}$$

$$\begin{aligned} & \times \int_{B_{\sigma(x)}(x)} \left( \sum_{k \in \mathbb{Z}} |\mathbf{a}_k(z)|^{r'} \mathbb{1}_{\Theta}(\theta - t\mathbf{c}_k(z)) \right)^{1/r'} W_t(y-z) dz \frac{dt}{t} dy dx \\ & \lesssim \int_{x \in \mathbb{R}} W_s(x) \int_{y \in B_s} \int_{t=\sigma(y)}^{(1x-2/L)\sigma(x)} \frac{t}{2\sigma(x)} W_t(y-x) h(y,t) \mathfrak{M}(y, \theta t^{-1}, t) \frac{dt}{t} dy dx \end{aligned}$$

Since the inmost integral vanishes unless  $|y-x| > 2\sigma_\theta(x)$ , we have that

$$\left( \int_{t=\sigma_\theta(y)}^{(1-2/L)\sigma_\theta(x)} \left| \frac{t}{2\sigma(x)} W_t(y-x) \right|^2 \frac{dt}{t} \right)^{1/2} \lesssim W_{\sigma(x)}(y-x)$$

it follows that

$$\begin{aligned} II & \lesssim \lambda \int_{x \in \mathbb{R}} W_s(x) \int_{y \in B_s} W_{\sigma(x)}(y-x) \left( \int_{t=0}^s |h(y,t)|^2 \frac{dt}{t} \right)^{1/2} dy dx \\ & \lesssim \lambda \int_{x \in \mathbb{R}} W_s(x) M \left( \left( \int_{t=0}^s |h(\cdot, t)|^2 \frac{dt}{t} \right)^{1/2} \right) (x) dx \\ & \lesssim \frac{\lambda \|h\|_{L^2_{(dydt/t)}}}{s^{1/2}} \end{aligned}$$

This concludes the proof.  $\square$

## 2.6 The energy embedding and non-iterated bounds

### 2.6.1 The energy embedding

Here we comment on how to deduce Theorem 2.2 from the result in [DPO15]. Let us fix a  $p \in (1, \infty]$  and  $q \in (\max(2; p'), +\infty]$  and without loss of generality let us suppose that  $\hat{f} \in C_c^\infty(\mathbb{R})$ . We will show that the weak versions of (2.11) holds i.e.

$$\|F\|_{L^{p, \infty} E^q(S_e)} \lesssim \|f\|_{L^p}. \quad (2.93)$$

By interpolation this would allow us to conclude the strong bounds of (2.11).

The paper [DPO15] deals with embeddings into the space  $\mathcal{X}$  that they denote by  $\mathcal{Z}$ . The generating collection of tents that they make use of is described in Section 2.1.2 of that paper. Notice that the set of geometric parameters for the tents in the present paper (Section 2.2) is larger than the one in [DPO15] but a careful perusal of the proofs therein shows that the same statements hold for the extended range of parameters.

Let us recall the main statements from [DPO15].

**Theorem 2.28** (Theorem 1 of [DPO15]). *Let  $f \in S(\mathbb{R})$  with  $\hat{f} \in C_c^\infty$ . Let  $p \in (1, 2)$  and consider the set  $\mathcal{I}_{f, \lambda, p}$  of maximal dyadic intervals contained in*

$$\mathcal{K}_{f, \lambda, p} = \{x \in \mathbb{R} : M_p f(x) > \lambda\} \quad \text{and let} \quad K_{f, \lambda, p} := \bigcup_{B_\tau(\zeta) \in \mathcal{I}_{f, \lambda, p}} D(\zeta, 3\tau). \quad (2.94)$$

Then with  $q \in (p', \infty]$ .

$$\|F \mathbb{1}_{\mathcal{X} \setminus K_{f, \lambda, p}}\|_{L^q(S_e)} \lesssim_{q, p} \lambda^{1-p/q} \|f\|_{L^p}^{p/q}.$$

We used the super level set of  $M_p f$  instead of the super level set of  $M_p(Mf)$  to define  $\mathcal{K}_{f,\lambda,p}$ . As mentioned in section 7.3.1 of [DPO15], the inner maximal function appears only in the reduction from the case with  $\widehat{f}$  compactly supported to the case with a general  $f \in S(\mathbb{R})$ . By our assumptions we can effectively ignore this complication.

**Proposition 2.29** (Proposition 3.2 + equations (2.6) and (2.7) of [DPO15]). *The estimate*

$$\|F \mathbb{1}_{D(x,s)}\|_{L^q(S_e)} \lesssim_{N,q} \left(1 + \frac{\text{dist}(\text{spt } f; B_s(x))}{s}\right)^{-N} \|f\|_{L^q}$$

holds for all  $N > 0$  and  $q \in (2, \infty]$ .

**Lemma 2.30** (Equation (7.3) of [DPO15]). *The estimate*

$$\|F \mathbb{1}_{D(x,s)}\|_{L^\infty(S_e)} \lesssim_N \left(1 + \frac{\text{dist}(\text{spt } f; B_s(x))}{s}\right)^{-N} \inf_{z \in B_s(x)} Mf(z)$$

holds for any  $N > 0$ .

**Corollary 2.31.** *Suppose that  $\text{spt } f \cap B_{2s}(x) = \emptyset$  then*

$$\|F \mathbb{1}_{D(x,s)}\|_{L^q(S_e)} s^{-1/q} \lesssim_{N,p} \left(1 + \frac{\text{dist}(\text{spt } f; B_s(x))}{s}\right)^{-N} s^{-1/p} \|f\|_{L^p}$$

for all  $p \in [1, 2)$ ,  $q > p'$ , and  $N > 0$ .

*Proof.* If  $\text{spt } f \cap B_{2s}(x) = \emptyset$  then  $\inf_{z \in B_s(x)} Mf(z) \lesssim s^{-1} \|f\|_{L^1}$ . Using this fact and interpolating between the bounds from Proposition 2.29 and Lemma 2.30 we obtain the required inequality.  $\square$

Fix  $p \in (1, \infty]$  and  $q \in (\max(p'; 2), \infty]$  and let  $\bar{p} \in (1, \min(p; 2))$  such that  $q > \bar{p}'$ . We will now show that

$$\|F \mathbb{1}_{\mathcal{X} \setminus K_{f,\lambda,\bar{p}}}\|_{L^q(S_e)} \lesssim \lambda. \quad (2.95)$$

Since  $\nu(K_{f,\lambda,\bar{p}}) \lesssim \lambda^{-p} \|f\|_{L^p}^p$  this would prove (2.93).

Let us consider a strip  $D(x, s) \in \mathbb{D}$  and suppose that  $D(x, s) \not\subset K_{f,\lambda,\bar{p}}$ , otherwise the estimate is trivial. We have  $B_{5s}(x) \not\subset K_{f,\lambda,\bar{p}}$ . For an  $N > 1$  large enough to be chosen later let us set

$$f(x) = f_0(x) + \sum_{k=1}^{\infty} f_k(x) = f_0(x) v\left(\frac{x-x_0}{5s}\right) + \sum_{k=1}^{\infty} f(x) \gamma\left(\frac{x-x_0}{5s2^{Nk}}\right)$$

where  $\gamma(\cdot) = v(\cdot/2^N) - v(\cdot)$  with

$$v \in C_c^\infty(B_2) \quad v \geq 0 \quad v = 1 \text{ on } B_1.$$

Let  $F_k$  be associated to  $f_k$  via the embedding (2.6) and let  $K_{f_k,\lambda,\bar{p}}$  be as in (2.94). Since  $K_{f_0,\lambda,\bar{p}} \subset K_{f,\lambda,\bar{p}}$  we have that  $\|f_0\|_{L^{\bar{p}} S^{-1/\bar{p}}} \lesssim \lambda$  and

$$\|F_0 \mathbb{1}_{\mathcal{X} \setminus K_{f,\lambda,\bar{p}}} \mathbb{1}_{D(x,s)}\|_{L^q(S_e)} \lesssim \lambda^{1-\bar{p}/q} \|f_0\|_{L^{\bar{p}}}^{\bar{p}/q} \lesssim \lambda s^{1/q} \quad (2.96)$$

by Theorem 2.28



Since  $K_{f_k, \lambda, \bar{p}} \subset K_{f, \lambda, \bar{p}} \not\supset B_{5s}(x)$  one has  $\|f_k\|_{L^{\bar{p}}} \lesssim \lambda \nu(D(x, s))^{1/\bar{p}} 2^{Nk/\bar{p}}$  and by Corollary 2.31 we have that

$$\|F_k \mathbb{1}_{\mathbb{X} \setminus K_{f, \lambda, \bar{p}}} \mathbb{1}_{D(x, s)}\|_{L^q(S_e)} s^{-1/q} \lesssim 2^{-2Nk} 2^{Nk/\bar{p}} \lambda \lesssim 2^{-Nk} \lambda. \quad (2.97)$$

By quasi-subadditivity we can add up (2.96) and (2.97) to obtain

$$\frac{\|F \mathbb{1}_{\mathbb{X} \setminus K_{f, \lambda, \bar{p}}} \mathbb{1}_{D(x, s)}\|_{L^q(S_e)}}{\nu(D(x, s))^{1/q}} \lesssim \lambda.$$

Since  $D(x, s)$  is arbitrary this implies (2.95).

### 2.6.2 Non-iterated bounds

We conclude by explaining that for  $r \in (2, \infty]$  and  $p \in (2, r)$  and simpler embedding bounds on the maps  $f \mapsto F$  and  $a \mapsto \mathbb{A}$  are sufficient to prove boundedness on  $L^p(\mathbb{R})$  of the Variational Carleson Operator (2.2) and thus also (2.1).

Hereafter we work with the non-iterated outer measure space  $(\mathbb{X}, \mu)$ . The energy embedding map satisfies the  $L^p$  bounds

$$\|F\|_{L^p(S_e)} \lesssim \|f\|_{L^p} \quad p \in (2, \infty]. \quad (2.98)$$

This follows directly from Proposition 2.29 by taking  $s$  arbitrarily large.

Similarly, in Proposition 2.20 we have shown that the auxiliary embedding satisfies

$$\|\mathbb{M}\|_{L^{p'}(S_m)} \lesssim \|\mathfrak{a}\|_{L^{p'}(I^{r'})} \quad p' \in (r', \infty]. \quad (2.99)$$

and thus, by Proposition 2.23 we have that the variational mass embedding also satisfies such bounds:

$$\|\mathbb{A}\|_{L^{p'}(S_m)} \lesssim \|\mathfrak{a}\|_{L^{p'}(I^{r'})} \quad p' \in (r', \infty]. \quad (2.100)$$

It follows by the outer Hölder inequality 2.7 that

$$\left| \iiint_{\mathbb{X}} F(y, \eta, t) \mathbb{A}(y, \eta, t) dy d\eta dt \right| \lesssim \|F\|_{L^p(S_e)} \|\mathbb{A}\|_{L^{p'}(S_m)}.$$

Using (2.98) and (2.100) and the wave-packet domination (2.13) it follows that (2.2) is bounded on  $L^p(\mathbb{R})$ .

In conclusion we remark that the iterated outer-measure  $L^p$  spaces that were introduced provide an effective way of capturing the spatial locality property of the embedding maps. Both the proof of Theorem 2.3 and of Theorem 2.2 rely on first obtaining non-iterated bounds (see Propositions 2.20 and 2.29) and then using a locality lemma (see Lemmata 2.21 and 2.30) and a projection lemma (see Lemma 2.22 and Lemma 7.8 of [DPO15]) to bootstrap the full result.



## Chapter 3

# Positive sparse domination of variational Carleson operators

Life, as we know it, is based on  $L^2$ .  
–F.D.P.

This Chapter contains the paper [DPDU16] Due to its nonlocal nature, the  $r$ -variation norm Carleson operator  $C_r$  does not yield to the sparse domination techniques of Lerner [Ler16; Ler13], Di Plinio and Lerner [DPL14], Lacey [Lac17]. We overcome this difficulty and prove that the dual form to  $C_r$  can be dominated by a positive sparse form involving  $L^p$  averages. Our result strengthens the  $L^p$ -estimates by Oberlin et. al. [OSTTW12]. As a corollary, we obtain quantitative weighted norm inequalities improving on [DL12a] by Do and Lacey. Our proof relies on the localized outer  $L^p$ -embeddings of Di Plinio and Ou [DPO15] and Uraltsev [Ura16].

### 3.1 Introduction and main results

The technique of controlling Calderón-Zygmund singular integrals, which are a-priori *non-local*, by *localized* positive sparse operators has recently emerged as a leading trend in Euclidean Harmonic Analysis. We briefly review the advancements which are most relevant for the present article and postpone further references to the body of the introduction. The original domination in norm result of [Ler13] for Calderón-Zygmund operators has since been upgraded to a *pointwise* positive sparse domination by Conde and Rey [CAR16] and Lerner and Nazarov [LN15], and later by Lacey [Lac17] by means of an inspiring stopping time argument forgoing local mean oscillation. Lacey’s approach was further clarified in [Ler16], resulting in the following principle: if  $T$  is a sub-linear operator of weak-type  $(p, p)$  and in addition the maximal operator

$$f \mapsto \sup_{Q \subset \mathbb{R} \text{ interval}} \|T(f \mathbb{1}_{\mathbb{R} \setminus 3Q})\|_{L^\infty(Q)} \mathbb{1}_Q \quad (3.1)$$

embodying the *non-locality* of  $T$ , is of weak-type  $(s, s)$ , for some  $1 \leq p \leq s < \infty$ , then  $T$  is pointwise dominated by a positive sparse operator involving  $L^s$  averages of  $f$ .

The principle (3.1) extends to certain modulated singular integrals. Of interest for us is the maximal partial Fourier transform

$$Cf(x) = \sup_N \left| \int_{-\infty}^N \widehat{f}(\xi) e^{ix\xi} d\xi \right|$$

also known as Carleson's operator on the real line. The crux of the matter is that (3.1) follows for  $T = C$  from its representation as a maximally modulated Hilbert transform, a fact already exploited in the classical weighted norm inequalities for  $C$  by Hunt and Young [HY74], and in the more recent work [GMS05]. Together with sharp forms of the Carleson-Hunt theorem near the endpoint  $p = 1$  [DP14] this allows, as observed by the first author and Lerner in [DPL14], the domination of  $C$  by sparse operators and thus leads to sharp weighted norm inequalities for  $C$ .

In this article we consider the  $r$ -variation norm Carleson operator, which is defined for Schwartz functions on the real line as

$$C_r f(x) = \sup_{N \in \mathbb{N}} \sup_{\xi_0 < \dots < \xi_N} \left( \sum_{j=1}^N \left| \int_{\xi_{j-1}}^{\xi_j} \widehat{f}(\xi) e^{ix\xi} d\xi \right|^r \right)^{1/r}.$$

The importance of  $C_r$  is revealed by the transference principle, presented in [OSTTW12, Appendix B], which shows how  $r$ -variational convergence of the Fourier series of  $f \in L^p(\mathbb{T}; w)$  for a weight  $w$  on the torus  $\mathbb{T}$  follows from  $L^p(\mathbb{R}; w)$ -estimates for the sub-linear operator  $C_r$ . Values of interest for  $r$  are  $2 < r < \infty$ . Indeed the main result of [OSTTW12] is that in this range,  $C_r$  maps into  $L^p$  whenever  $p > r'$ , while no  $L^p$ -estimates hold for variation exponents  $r \leq 2$ . Unlike the Carleson operator, its variation norm counterpart  $C_r$  does not have an explicit kernel form and thus fails to yield to Hunt-Young type techniques. The same essential difficulty is encountered in the search for  $L^q$ -bounds for the nonlocal maximal function (3.1) when  $T = C_r$ . Therefore, the approach via (3.1) does not seem to be applicable to  $C_r$ . In the series [DL12a; DL12b], the second author and Lacey circumvented this issue through a direct proof of  $A_p$ -weighted inequalities for  $C_r$  and its Walsh analogue, based on weighted phase plane analysis.

The main result of the present article is that a sparse domination principle for  $C_r$  holds in spite of the difficulties described above. More precisely, we sharply dominate the dual form to the  $r$ -variational Carleson operator  $C_r$  by a single positive sparse form involving  $L^p$  averages, leading to an effortless strengthening of the weighted theory of [DL12a]. Our argument abandons (3.1) in favor of a stopping time construction, relying on the localized Carleson embeddings for suitably modified wave packet transforms of [DPO15] by the first author and Yumeng Ou, and [Ura16] by the third author. In particular, our technique requires no *a-priori* weak-type information on the operator  $T$ . A similar approach was employed by Culiuc, Ou and the first author in [CDPO16] in the proof of a sparse domination principle for the family of modulation invariant multi-linear multipliers whose paradigm is the bilinear Hilbert transforms. Interestingly, unlike [CDPO16], our construction of the sparse collection in Section 3.4 seems to be the first in literature which does not make any use of dyadic grids.

We believe that intrinsic sparse domination can prove useful in the study of other classes of multi-linear operators lying way beyond the scope of Calderón-Zygmund theory, such as the iterated Fourier integrals of [DMT17] and the sub-dyadic multipliers of [BB17].

To formulate our main theorem, we recall the notation

$$\langle f \rangle_{I,p} := \left( \frac{1}{|I|} \int |f|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

where  $I \subset \mathbb{R}$  is any interval, and the notion of a *sparse collection* of intervals. We say that the countable collection of intervals  $I \in \mathcal{S}$  is  $\eta$ -sparse for some  $0 < \eta \leq 1$  if there exist a choice of measurable sets  $\{E_I : I \in \mathcal{S}\}$  such that

$$E_I \subset I, \quad |E_I| \geq \eta|I|, \quad E_I \cap E_J = \emptyset \quad \forall I, J \in \mathcal{S}, I \neq J.$$

**Theorem 3.1.** *Let  $2 < r < \infty$  and  $p > r'$ . Given  $f, g \in C_0^\infty(\mathbb{R})$  there exists a sparse collection  $\mathcal{S} = \mathcal{S}(f, g, p)$  and an absolute constant  $K = K(p)$  such that*

$$|\langle C_r f, g \rangle| \leq K(p) \sum_{I \in \mathcal{S}} |I| \langle f \rangle_{I,p} \langle g \rangle_{I,1}. \quad (3.2)$$

A corollary of Theorem 3.1 is that  $C_r$  extends to a bounded sub-linear operator on  $L^q(\mathbb{R})$  whenever  $q > r'$ . As a matter of fact, let us fix  $q \in (r', \infty]$ , and choose  $p \in (r', q)$ . Denoting by

$$M_p f(x) = \sup_{I \ni x} \langle f \rangle_{I,p}$$

the  $p$ -th Hardy-Littlewood maximal function, the estimate of Theorem 3.1 and the fact that  $\mathcal{S}$  is sparse yields

$$|\langle C_r f, g \rangle| \lesssim \sum_{I \in \mathcal{S}} |E_I| \langle f \rangle_{I,p} \langle g \rangle_{I,1} \leq \langle M_p f, M_1 g \rangle \lesssim \|M_p f\|_q \|M_1 g\|_{q'} \lesssim \|f\|_q \|g\|_{q'}.$$

Bounds on  $L^q$  for  $C_r$  were first proved in [OSTTW12], where it is also shown that the restriction  $q > r'$  is necessary, whence no sparse domination of the type occurring in Theorem 3.1 will hold for  $p < r'$ . We can thus claim that Theorem 3.1 is sharp, short of the endpoint  $p = r'$ . In fact, sparse domination as in (3.2) also entails  $C_r : L^p(\mathbb{R}) \rightarrow L^{p,\infty}(\mathbb{R})$ . Such an estimate is currently unknown for  $p = r'$ .

However, Theorem 3.1 yields much more precise information than mere  $L^q$ -boundedness. In particular, we obtain precisely quantified weighted norm inequalities for  $C_r$ . Recall the definition of the  $A_t$  constant of a locally integrable nonnegative function  $w$  as

$$[w]_{A_t} := \begin{cases} \sup_{I \subset \mathbb{R}} \langle w \rangle_{I,1} \langle w^{\frac{1}{1-t}} \rangle_{I,1}^{t-1} & 1 < t < \infty \\ \inf \{ A : Mw(x) \leq Aw(x) \text{ for a.e. } x \} & t = 1 \end{cases}$$

**Theorem 3.2.** *Let  $2 < r < \infty$  and  $q > r'$  be fixed. Then*

(i) *there exists  $K : [1, \frac{q}{r'}) \rightarrow (0, \infty)$  nondecreasing such that*

$$\|C_r\|_{L^q(\mathbb{R};w) \rightarrow L^q(\mathbb{R};w)} \leq K(t) [w]_{A_t}^{\max\{1, \frac{t}{q(t-1)}\}};$$

(ii) *there exists a positive increasing function  $\mathcal{Q}$  such that for  $t = \frac{q}{r'}$*

$$\|C_r\|_{L^q(\mathbb{R};w) \rightarrow L^q(\mathbb{R};w)} \leq \mathcal{Q}([w]_{A_t}). \quad (3.3)$$

We omit the standard deduction of Theorem 3.2 from Theorem 3.1, which follows along lines analogous to the proofs of [CDPO16, Theorem 3] and [LN15, Theorem 17.1]. Estimate (i) of Theorem 3.2 yields in particular that

$$w \in A_t \implies \|C_r\|_{L^q(\mathbb{R};w) \rightarrow L^q(\mathbb{R};w)} < \infty \quad \forall r > \max\left\{2, \frac{q}{q-t}\right\}$$

an improvement over [DL12a, Theorem 1.2], where  $L^q(\mathbb{R};w)$  boundedness is only shown for variation exponents  $r > \max\left\{2t, \frac{qt}{q-t}\right\}$  when  $w \in A_t$ . Fixing  $r$  instead, part (ii) of Theorem 3.2 is sharp in the sense that  $t = \frac{q}{r'}$  is the largest exponent such that an estimate of the type of (3.3) is allowed to hold. Indeed, if (3.3) were true for any  $q = q_0 \in (r', \infty)$  and some  $t = \frac{q_0}{s}$

with  $s < r'$ , a version of the Rubio de Francia extrapolation theorem (see for instance [CUMP11, Theorem 3.9]) would yield that  $C_r$  maps  $L^q$  into itself for all  $q \in (s, \infty)$ , contradicting the already mentioned counterexample from [OSTTW12].

We turn to further comments on the proof and on the structure of the paper. In the upcoming Section 3.2 we reduce the bilinear form estimate (3.2) to an analogous statement for a bilinear form involving integrals over the upper-three space of symmetry parameters for the Carleson operator of a wave packet transforms of  $f$  and a variational-truncated wave packet transform of  $g$ . The natural framework for  $L^p$ -boundedness of such forms, the  $L^p$ -theory of outer measures, has been developed by the second author and Thiele in [DT15]. In Section 3.3, we recall the basics of this theory as well as the localized Carleson embeddings of [DPO15] and [Ura16]. These will come to fruition in Section 3.4, where we give the proof of Theorem 3.1. A significant challenge in the course of the proof is the treatment of the nonlocal (tail) components, which are handled via novel *ad-hoc* embedding theorems incorporating the fast decay of the wave packet coefficients away from the support of the input functions.

## Acknowledgments

This work was initiated and continued during G. Uraltsev's visit to the Brown University and University of Virginia Mathematics Departments, whose hospitality is gratefully acknowledged. The authors would like to thank Amalia Culiuc, Michael Lacey, Ben Krause and Yumeng Ou for useful conversations about sparse domination principles.

## 3.2 Reduction to wave packet transforms

In this section we reduce the inequality (3.2) to an analogous statement involving wave packet transforms. Throughout this section, the variation exponent  $r \in (2, \infty)$  is fixed, and we take  $f, g \in C_0^\infty(\mathbb{R})$ . First of all we linearize the variation norm appearing in  $C_r$ . Begin by observing that the map

$$(x, \xi) \mapsto \int_{-\infty}^{\xi} \widehat{f}(\zeta) e^{ix\zeta} d\zeta$$

is uniformly continuous. By duality and standard considerations

$$C_r f(x) = \sup_N \sup_{\Xi \subset \mathbb{R}, \#\Xi \leq N} \sup_{\|\{a_j\}\|_{\ell^{r'}} \leq 1} \sum_{j=1}^N a_j \int_{\xi_{j-1}}^{\xi_j} \widehat{f}(\zeta) e^{ix\zeta} d\zeta.$$

Therefore, (3.2) will be a consequence of the estimate

$$\Lambda_{\vec{\xi}, \vec{a}}(f, g) := \int_{\mathbb{R}} g(x) \left( \sum_{j=1}^N a_j(x) \int_{\xi_{j-1}(x)}^{\xi_j(x)} \widehat{f}(\zeta) e^{ix\zeta} d\zeta \right) dx \leq K(p) \sum_{I \in \mathcal{S}} |I| \langle f \rangle_{I,p} \langle g \rangle_{I,1} \quad (3.4)$$

with right hand side independent of  $N \in \mathbb{N}$ ,  $\Xi \subset \mathbb{R}$ ,  $\#\Xi \leq N$ , and of the measurable  $\Xi^{N+1}$ -valued function  $\vec{\xi}(x) = \{\xi_j(x)\}$  with  $\xi_0(x) < \dots < \xi_N(x)$ , and  $\mathbb{C}^{N+1}$ -valued  $\vec{a}(x) = \{a_j(x)\}$  with  $\|\vec{a}(x)\|_{\ell^{r'}} = 1$ .

The next step is to uniformly dominate the form  $\Lambda_{\vec{\xi}, \vec{a}}(f, g)$  by an outer form involving wave packet transforms of  $f$  and  $g$ ; in the terminology of [DT15], *embedding* maps into the upper 3-space

$$(u, t, \eta) \in \mathcal{X} = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}.$$

The parameters  $\xi, \vec{a}$  will enter the definition of the embedding map for  $g$ . We introduce the wave packets

$$\psi_{t,\eta}(x) := t^{-1} e^{i\eta z} \psi\left(\frac{x}{t}\right), \quad \eta \in \mathbb{R}, t \in (0, \infty)$$

where  $\psi$  is a real valued, even Schwartz function with frequency support of width  $b$  containing the origin. The wave packet transform of  $f$  is thus defined, as in [DT15], by

$$F(f)(u, t, \eta) = |f * \psi_{t,\eta}(u)|, \quad (u, t, \eta) \in \mathcal{X}. \quad (3.5)$$

For our fixed choice of  $\vec{\xi}, \vec{a}$  we introduce the modified wave packet transform of  $g$  that is dual to (3.5) for the sake of bounding the left hand side of (3.4). Following [Ura16, Eq. (1.14)], it is given by

$$A(g)(u, t, \eta) := \sup_{\Psi} \left| \int_{\mathbb{R}} g(x) \sum_{j=1}^N a_j(x) \Psi_{t,\eta}^{\xi_j(x), \xi_{j+1}(x)}(x-u) dx \right|, \quad (u, t, \eta) \in \mathcal{X}, \quad (3.6)$$

with supremum being taken over all choices of *truncated wave packets*  $\Psi_{t,\eta}^{\xi_-, \xi_+}$ , that for each  $t, \eta \in \mathbb{R}_+ \times \mathbb{R}$  are functions in  $\mathcal{S}(\mathbb{R})$  parameterized by  $\xi_-, \xi_+ \in \Xi$ . We summarize the basic defining properties of the truncated wave packets in Remark 3.5 below, and we refer to [Ura16] for a precise definition.

The duality of the embeddings (3.5) and (3.6) is a consequence of the following wave packet domination Lemma. We send to [Ura16] for the proof.

**Lemma 3.3** (Wave packet domination). *Let  $f, g, \Xi, \vec{\xi}, \vec{a}$  be as above and consider the bilinear form defined on the wave packets transforms, given by*

$$\mathbb{B}_{\vec{\xi}, \vec{a}}(f, g) := \int_{\mathcal{X}} F(f)(u, t, \eta) A(g)(u, t, \eta) du dt d\eta. \quad (3.7)$$

Then

$$\Lambda_{\vec{\xi}, \vec{a}}(f, g) \lesssim \mathbb{B}_{\vec{\xi}, \vec{a}}(F(f), A(g))$$

with uniform implied constant.

Using the above Lemma, we see that inequality (3.4) and thus Theorem 3.1 will follow from the bounds of the next proposition.

**Proposition 3.4.** *Let  $p > r'$  be fixed. For all  $f, g \in L^\infty(\mathbb{R})$  and compactly supported there exists a sparse collection  $\mathcal{S} = \mathcal{S}(f, g, p)$  and an absolute constant  $K = K(p)$  such that*

$$\sup_N \sup_{\#\Xi \leq N} \sup_{\vec{\xi}, \vec{a}} \mathbb{B}_{\vec{\xi}, \vec{a}}(f, g) \leq K(p) \sum_{I \in \mathcal{S}} |I| \langle f \rangle_{I, p} \langle g \rangle_{I, 1} \quad (3.8)$$

where  $\vec{\xi}, \vec{a}$  range over  $\Xi^{N+1}, \mathbb{C}^{N+1}$ -valued functions as above.

We now make a brief digression to justify definitions (3.5) and (3.6) of the wave packet transforms and the result of Lemma 3.3. Consider the term

$$\int_{\xi_{j-1}(x)}^{\xi_j(x)} \widehat{f}(\eta) e^{ix\eta} d\eta$$

appearing in (3.4) and let us think for a moment of  $\xi_{j-1}(x) = \xi_-$  and  $\xi_j(x) = \xi_+$  as frozen. Then the following representation holds for the multiplier  $1_{(\xi_-, \xi_+)}(\zeta)$ :

$$1_{(\xi_-, \xi_+)}(\zeta) = \int_{\mathbb{R}_+ \times \mathbb{R}} \widehat{\Psi}_{t,\eta}^{\xi_-, \xi_+}(\zeta) dt d\eta \quad (3.9)$$

where  $\Psi_{t,\eta}^{\xi_-, \xi_+}$  are truncated wave packets. Choosing a  $\varphi \in \mathcal{S}(\mathbb{R})$  such that  $\widehat{\varphi}_{t,\eta}(\zeta) = 1$  whenever  $\widehat{\Psi}_{t,\eta}^{\xi_-, \xi_+}(\zeta) \neq 0$  for any  $\xi_- < \xi_+ \in \mathbb{R}$ , we obtain the pointwise identity

$$\int_{\xi_{j-1}(x)}^{\xi_j(x)} \widehat{f}(\zeta) e^{ix\zeta} d\zeta = \int_{\mathbb{X}} f * \varphi_{t,\eta}(u) \Psi_{t,\eta}^{\xi_{j-1}(x), \xi_j(x)}(x-u) du dt d\eta.$$

The results of Lemma 3.3 follow by Fubini and the triangle inequality.

We briefly illustrate identity (3.9), for a more careful discussion we refer to [Ura16, Sec. 3]. Start by choosing  $\psi \in \mathcal{S}(\mathbb{R})$  with  $\widehat{\psi}$  non-negative and supported on a ball of radius  $b/2$ , and let  $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$  be a non-negative bump function supported on  $[d - \varepsilon, d + \varepsilon]$  with  $d > b$  and  $\varepsilon \ll b$ . Suppose formally that  $\xi_+ = +\infty$  so that, up to a suitable normalization of  $\chi$ , a Littlewood-Paley type decomposition centered at  $\xi_-$  of the multiplier  $1_{(\xi_-, +\infty)}$  gives

$$1_{(\xi_-, +\infty)}(\zeta) = \int_{\mathbb{R}_+ \times \mathbb{R}} \widehat{\psi}(t(\zeta - \eta)) \chi(t(\eta - \xi_-)) dt d\eta.$$

A similar expression holds if  $\xi_- = -\infty$  and  $\xi_+ \in \mathbb{R}$ . We choose truncated wave packets so that

$$\begin{aligned} \Psi_{t,\eta}^{\xi_-, \xi_+}(x) &:= \psi_{t,\eta}(x) \chi(t(\eta - \xi_-)) & t(\eta - \xi_-) &\ll t(\xi_+ - \eta) \\ \Psi_{t,\eta}^{\xi_-, \xi_+}(x) &:= \psi_{t,\eta}(x) \chi(t(\xi_+ - \eta)) & t(\eta - \xi_-) &\gg t(\xi_+ - \eta) \\ \Psi_{t,\eta}^{\xi_-, \xi_+}(x) &:= 0 & \eta &\notin (\xi_-, \xi_+). \end{aligned}$$

Finally if  $t(\eta - \xi_-) \approx t(\xi_+ - \eta)$  then  $\Psi_{t,\eta}^{\xi_-, \xi_+}$  is chosen to appropriately model the transition between the above regimes and justifies identity (3.9).

**Remark 3.5.** In general we call a function  $\Psi_{t,\eta}^{\xi_-, \xi_+} \in \mathcal{S}(\mathbb{R})$  parameterized by  $\xi_- < \xi_+ \in \mathbb{R}$  a truncated wave-packet adapted to  $t, \eta \in \mathbb{R}_+ \times \mathbb{R}$  if

$$e^{-intz} t \Psi_{t,\eta}^{\xi_-, \xi_+}(tx), \quad t^{-1} \partial_{\xi_-} \left( e^{-intz} t \Psi_{t,\eta}^{\xi_-, \xi_+}(tx) \right), \quad t^{-1} \partial_{\xi_+} \left( e^{-intz} t \Psi_{t,\eta}^{\xi_-, \xi_+}(tx) \right)$$

are uniformly bounded in  $\mathcal{S}(\mathbb{R})$  as functions of  $x$ . Furthermore we require that  $\widehat{\Psi}_{t,\eta}^{\xi_-, \xi_+}$  be supported on  $(\eta - t^{-1}b, \eta + t^{-1}b)$  for some  $b > 0$ . Finally, for some constants  $d, d', d'' > 0$  and  $\varepsilon > 0$  it must hold that

$$\begin{aligned} \Psi_{t,\eta}^{\xi_-, \xi_+} \neq 0 & \quad \text{only if} \quad \begin{cases} t(\eta - \xi_-) \in (d - \varepsilon, d + \varepsilon) \\ t(\xi_+ - \eta) > d' > 0 \end{cases} & (3.10) \\ \partial_{\xi_+} \Psi_{t,\eta}^{\xi_-, \xi_+} = 0 & \quad \text{if} \quad t(\xi_+ - \eta) > d'' > d' > 0. \end{aligned}$$

### 3.3 Localized outer $L^p$ embeddings

We now turn to the description of the analytic tools which are relied upon in the proof of estimate (3.2). We work in the framework of outer measure spaces [DT15], see also [CDPO16; DPO15].



In particular, we define a distinguished collection of subsets of the upper 3-space  $\mathbb{X}$  which we refer to as *tents* above the time-frequency loci  $(I, \xi)$  where  $I$  is an interval of center  $c(I)$  and length  $|I|$ , and  $\xi \in \mathbb{R}$ :

$$\begin{aligned} \mathbb{T}(I, \xi) &:= \mathbb{T}^\ell(I, \xi) \cup \mathbb{T}^\circ(I, \xi), \\ \mathbb{T}^\circ(I, \xi) &:= \{(u, t, \eta) : t\eta - t\xi \in \Theta^\circ, t < |I|, |u - c(I)| < |I| - t\} \\ \mathbb{T}^\ell(I, \xi) &:= \{(u, t, \eta) : t\eta - t\xi \in \Theta \setminus \Theta^\circ, t < |I|, |u - c(I)| < |I| - t\} \end{aligned}$$

where  $\Theta^\circ = [\beta^-, \beta^+]$ ,  $\Theta = [\alpha^-, \alpha^+]$  are two geometric parameter intervals such that  $0 \in \Theta^\circ \subset \Theta$ . The specific values of the parameters do not matter. What is important that given the geometric parameters of the wave packets appearing in (3.5) and (3.6) there exists a choice of parameters of the tents such that the statements of the subsequent discussion hold. For example it must hold that  $(-b, b) \subset \Theta^\circ$  where  $b$  is the parameter that governs the frequency support of  $\psi_{t,\eta}$  and  $\Psi_{t,\eta}^{\xi^-, \xi^+}$ . For a complete discussion see [Ura16, Sec. 2]. As usual, we denote by  $\mu$  the outer measure generated by countable coverings by tents  $\mathbb{T}(I, \xi)$ ,  $I \subset \mathbb{R}$ ,  $\xi \in \mathbb{R}$  via the pre-measure  $\mathbb{T}(I, \xi) \mapsto |I|$ .

Let  $\mathfrak{s}$  be a size [DT15], i.e. a family of quasi-norms indexed by tents  $\mathbb{T}$ , defined on Borel functions  $F : \mathbb{X} \rightarrow \mathbb{C}$ . The corresponding outer- $L^p$  space on  $(\mathbb{X}, \mu)$  is defined by the quasi-norm

$$\begin{aligned} \|F\|_{L^p(\mathfrak{s})} &:= \left( p \int_0^\infty \lambda^{p-1} \mu(\mathfrak{s}(F) > \lambda) d\lambda \right)^{\frac{1}{p}}, \quad 0 < p < \infty, \\ \mu(\mathfrak{s}(F) > \lambda) &:= \inf \left\{ \mu(E) : E \subset \mathbb{X}, \sup_{\mathbb{T}} \mathfrak{s}(F \mathbb{1}_{\mathbb{X} \setminus E})(\mathbb{T}) \leq \lambda \right\} \end{aligned}$$

where the supremum on the right is taken over all tents  $\mathbb{T} = \mathbb{T}(I, \xi)$ . We will work with outer  $L^p$  spaces based on the sizes

$$\begin{aligned} \mathfrak{s}^\varepsilon(F)(\mathbb{T}) &:= \left( \frac{1}{|I|} \int_{\mathbb{T}^\ell} |F(u, t, \eta)|^2 dudtd\eta \right)^{\frac{1}{2}} + \sup_{(u, t, \eta) \in \mathbb{T}} |F(u, t, \eta)|, \\ \mathfrak{s}^m(A)(\mathbb{T}) &:= \left( \frac{1}{|I|} \int_{\mathbb{T}} |A(u, t, \eta)|^2 dudtd\eta \right)^{\frac{1}{2}} + \frac{1}{|I|} \int_{\mathbb{T}^\circ} |A(u, t, \eta)| dudtd\eta \end{aligned}$$

that are related to the two embeddings (3.5) and (3.6) respectively. The dual relation of the sizes  $\mathfrak{s}^\varepsilon, \mathfrak{s}^m$  is given by the fact that for any two Borel functions  $F, A : \mathbb{X} \rightarrow \mathbb{C}$  there holds

$$\int_{\mathbb{T}} |F(u, t, \eta)A(u, t, \eta)| dudtd\eta \leq 2\mathfrak{s}^\varepsilon(F)(\mathbb{T})\mathfrak{s}^m(A)(\mathbb{T}).$$

The abstract *outer Hölder inequality* [DT15, Prop. 3.4] and Radon-Nikodym type bounds [DT15, Prop. 3.6] yield

$$\int_{\mathbb{T}} |F(u, t, \eta)A(u, t, \eta)| dudtd\eta \lesssim \|F\|_{L^\sigma(\mathfrak{s}^\varepsilon)} \|A\|_{L^\tau(\mathfrak{s}^m)} \quad (3.11)$$

whenever  $1 \leq \sigma, \tau \leq \infty$  are Hölder dual exponents i.e.  $\frac{1}{\sigma} + \frac{1}{\tau} = 1$ .

The nature of the wave packet transforms  $f \mapsto F(f)$ ,  $g \mapsto A(g)$  defined by (3.5), (3.6) is heavily exploited in the stopping-type outer  $L^p$ -embedding theorems below. We state the embedding theorems after some necessary definitions. It is convenient to use the notation

$$\mathbb{T}(I) := \{(u, t, \eta) : t < |I|, |u - c(I)| < |I| - t, \eta \in \mathbb{R}\}$$

for the set of the upper 3-space associated to the usual spatial tent over  $I$ . Given an open set  $E \subset \mathbb{R}$  we associate to it the subset of  $\mathbb{T}(E) \subset \mathbb{X}$  given by

$$\mathbb{T}(E) = \bigcup_{I \subset E} \mathbb{T}(I) \quad (3.12)$$

where the union is taken over all *intervals*  $I \subset E$ .

The first stopping embedding theorem, a reformulation of a result first obtained in [DPO15], deals with the wave packet transform  $f \mapsto F(f)$  of (3.5).

**Proposition 3.6.** *Let  $1 < p < 2$ ,  $\sigma \in (p', \infty)$ , then there exists  $K > 0$  such that the following holds. For all  $f \in L^p_{\text{loc}}(\mathbb{R})$ , all intervals  $Q$ , and all  $c \in (0, 1)$  there exists an open set  $U_{f,p,Q}$  satisfying*

$$|U_{f,p,Q}| \leq c|Q|,$$

such that

$$\|F(f\mathbb{1}_{3Q})\mathbb{1}_{\mathbb{T}(Q)\setminus\mathbb{T}(U_{f,p,Q})}\|_{L^\sigma(s^e)} \leq K|Q|^{\frac{1}{\sigma}}\langle f \rangle_{3Q,p}. \quad (3.13)$$

The embedding theorem we use to treat the variationally truncated wave packet transform  $g \mapsto A(g)$  of (3.6) stems from the main result of [Ura16].

**Proposition 3.7.** *Let  $\tau \in (r', \infty)$ , then there exists  $K > 0$  such that the following holds. For all  $g \in L^1_{\text{loc}}(\mathbb{R})$ , all intervals  $Q$  and all  $c \in (0, 1)$  there exists an open set  $V_{g,1,Q}$  satisfying*

$$|V_{g,1,Q}| \leq c|Q|,$$

such that

$$\|A(g\mathbb{1}_{3Q})\mathbb{1}_{\mathbb{T}(Q)\setminus\mathbb{T}(V_{g,1,Q})}\|_{L^\tau(s^m)} \leq K|Q|^{\frac{1}{\tau}}\langle g \rangle_{3Q,1}. \quad (3.14)$$

We stress that the constant  $K$  in Proposition 3.7 does not depend on the parameters  $\vec{a}, \vec{\xi}, \vec{\Xi}, N$  appearing in the definition (3.6) of the map  $A$ .

The above two propositions appear in [Ura16] in a somewhat different form that uses the notion of iterated outer measure spaces introduced therein. We derive the statement of Propositions 3.6 and 3.7 by using the weak boundedness on  $L^1(\mathbb{R})$  of the map (3.6) of [Ura16, Theorem 1.3]. In particular that result, applied to the function  $g\mathbb{1}_{3Q}$  for  $\lambda = cK\langle g \rangle_{3Q,1}$ , yields a collection of disjoint open intervals  $\mathcal{I}$  and

$$V_{g,1,Q} := \bigcup_{I \in \mathcal{I}} I, \quad |V_{g,1,Q}| \leq \frac{C|Q|}{K}$$

so that (3.14) holds as required. We conclude by choosing  $K \geq C/c$ . A similar procedure can be used to obtain Proposition 3.6 from [Ura16, Theorem. 1.2].

In effect, we have shown that the formulation of the boundedness properties of the embedding maps (3.5) and (3.6) as expressed in Propositions 3.6 and 3.7 are equivalent to the iterated outer measure formulation of [Ura16]. Furthermore the use of iterated outer measure  $L^p$  norms allowed us to bootstrap the above results to  $L^p_{\text{loc}}(\mathbb{R})$  generality from an a-priori type statement, as illustrated in [Ura16, Section 2.1].

### 3.4 Proof of Proposition 3.4

Throughout this proof, the exponent  $p \in (r', \infty)$  is fixed and all the implicit constants are allowed to depend on  $r, p$  without explicit mention. Since the linearization parameters play no explicit

role in the upcoming arguments we omit them from the notation, assume them fixed and simply write  $B(f, g)$  for the form  $B_{\xi, \bar{a}}(f, g)$  defined in (3.7). Given any interval  $Q$ , we introduce the localized version

$$B_Q(f, g) := \int_{\mathbb{T}(Q)} F(f)(u, t, \eta) A(g)(u, t, \eta) \, dudtd\eta. \quad (3.15)$$

### 3.4.1 The principal iteration

The main step of the proof of Proposition 3.4 is contained in the following lemma, which we will apply iteratively.

**Lemma 3.8.** *There exists a positive constant  $K$  such that the following holds. Let  $f, g \in L^\infty(\mathbb{R})$  and compactly supported, and  $Q \subset \mathbb{R}$  be any interval. There exists a countable collection of disjoint open intervals  $\mathcal{I}_Q$  such that*

$$\bigcup_{I \in \mathcal{I}_Q} I \subset Q, \quad \sum_{I \in \mathcal{I}_Q} |I| \leq 2^{-12} |Q| \quad (3.16)$$

and such that

$$B_Q(f \mathbb{1}_{3Q}, g \mathbb{1}_{3Q}) \leq K |Q| \langle f \rangle_{3Q, p} \langle g \rangle_{3Q, 1} + \sum_{I \in \mathcal{I}_Q} B_I(f \mathbb{1}_{3I}, g \mathbb{1}_{3I}). \quad (3.17)$$

The proof of the lemma consists of several steps, which we now begin. Notice that there is no loss in generality with assuming that  $f, g$  are supported on  $3Q$ : we do so for mere notational convenience.

#### Construction of $\mathcal{I}_Q$

Referring to the notations of Section 3.3, set

$$\begin{aligned} E_{f, Q} &= U_{f, p, Q} \cup \{x \in \mathbb{R} : M_p f(x) > c^{-1} \langle f \rangle_{3Q, p}\}, \\ E_{g, Q} &= V_{g, 1, Q} \cup \{x \in \mathbb{R} : M_1 g(x) > c^{-1} \langle g \rangle_{3Q, 1}\}, \\ E_Q &= Q \cap (E_{f, Q} \cup E_{g, Q}). \end{aligned}$$

Write the open set  $E_Q$  as the union of a countable collection  $I \in \mathcal{I}_Q$  of disjoint open intervals. Then (3.16) holds provided that  $c$  is chosen small enough. Also, necessarily  $3I \cap E_Q^c \neq \emptyset$  if  $I \in \mathcal{I}_Q$ , so that

$$\inf_{x \in 3I} M_1 f(x) \lesssim \langle f \rangle_{3Q, p}, \quad \inf_{x \in 3I} M_1 g(x) \lesssim \langle g \rangle_{3Q, 1}. \quad (3.18)$$

For further use we note that, with reference to the notations of Propositions 3.6 and 3.7,

$$\mathbb{T}(Q) \setminus \mathbb{T}(E_Q) \subset \mathbb{T}(Q) \setminus (\mathbb{T}(U_{f, p, Q}) \cup \mathbb{T}(V_{g, 1, Q})) \quad (3.19)$$

This completes the construction of  $\mathcal{I}_Q$ .

#### Proof of (3.17)

We begin by using (3.12) to partition the outer integral over  $\mathbb{T}(Q)$  as

$$B_Q(f, g) \leq \int_{\mathbb{T}(Q) \setminus \mathbb{T}(E_Q)} F(f) A(g) \, dudtd\eta + \sum_{I \in \mathcal{I}_Q} B_I(f, g) \quad (3.20)$$

Choosing  $\tau \in (r', p)$ , the dual exponent  $\sigma = \tau' \in (p', \infty)$ . By virtue of (3.19), we may apply the outer Hölder inequality (3.11) and the embeddings Propositions 3.6 and 3.7 to control the first summand in (3.20) by an absolute constant times

$$\|F(f)\mathbb{1}_{\mathbb{T}(Q)\setminus\mathbb{T}(U_{f,p,Q})}\|_{L^\sigma(\mathfrak{s}^\ell)} \|A(g)\mathbb{1}_{\mathbb{T}(Q)\setminus\mathbb{T}(V_{g,1,Q})}\|_{L^\tau(\mathfrak{s}^o)} \lesssim |Q|\langle f \rangle_{3Q,p} \langle g \rangle_{3Q,1}.$$

We turn to the second summand in (3.20), which is less than or equal to

$$\sum_{I \in \mathcal{I}_Q} \mathbf{B}_I(f\mathbb{1}_{3I}, g\mathbb{1}_{3I}) + \sum_{\substack{(\mathbf{a}, \mathbf{b}) \in \{\text{in}, \text{out}\}^2 \\ (\mathbf{a}, \mathbf{b}) \neq (\text{in}, \text{in})}} \sum_{I \in \mathcal{I}_Q} \mathbf{B}_I(f\mathbb{1}_{I^{\mathbf{a}}}, g\mathbb{1}_{I^{\mathbf{b}}}).$$

where  $I^{\text{in}} = 3I, I^{\text{out}} = 3Q \setminus 3I$ . The first term in the above display appears on the right hand side of (3.17). We claim that

$$\sum_{I \in \mathcal{I}_Q} \mathbf{B}_I(f\mathbb{1}_{I^{\mathbf{a}}}, g\mathbb{1}_{I^{\mathbf{b}}}) \lesssim |Q|\langle f \rangle_{3Q,p} \langle g \rangle_{3Q,1}, \quad (\mathbf{a}, \mathbf{b}) \neq (\text{in}, \text{in}) \quad (3.21)$$

thus leading to the required estimate for (3.17). Assume  $\mathbf{a} = \text{in}, \mathbf{b} = \text{out}$  for the sake of definiteness, the other cases being identical. Fix  $I \in \mathcal{I}_Q$ . We will show that

$$\mathbf{B}_I(f\mathbb{1}_{I^{\text{in}}}, g\mathbb{1}_{I^{\text{out}}}) \lesssim |I|\langle f \rangle_{3Q,p} \langle g \rangle_{3Q,1}. \quad (3.22)$$

whence (3.21) follows by summing over  $I \in \mathcal{I}_Q$  and taking advantage of (3.16).

### Proof of (3.22)

We introduce the Carleson box over the interval  $P$

$$\text{box}(P) = \{(u, t, \eta) \in \mathbb{X} : u \in P, \frac{1}{2}|P| \leq t < |P|\}$$

Fix  $I \in \mathcal{I}_Q$ . At the root of our argument for (3.22) is the fact that  $\text{supp } g\mathbb{1}_{I^{\text{out}}}$  lies outside  $3I$ . This leads to the exploitation of the following lemma, whose proof is given at the end of the paragraph.

**Lemma 3.9.** *Let  $P$  be any interval,  $h \in L^p_{\text{loc}}(\mathbb{R})$ , and  $\tau, \sigma$  as above. There holds*

$$\|A(h)\mathbb{1}_{\text{box}(P)}\|_{L^\tau(\mathfrak{s}^m)} \lesssim |P|^{\frac{1}{\tau}} \left(1 + \frac{\text{dist}(P, \text{supp } h)}{|P|}\right)^{-100} \inf_{x \in 3P} M_1 h(x), \quad (3.23)$$

$$\|F(h)\mathbb{1}_{\text{box}(P)}\|_{L^\sigma(\mathfrak{s}^e)} \lesssim |P|^{\frac{1}{\sigma}} \left(1 + \frac{\text{dist}(P, \text{supp } h)}{|P|}\right)^{-100} \inf_{x \in 3P} M_p h(x). \quad (3.24)$$

Now let  $P \in \mathcal{P}_k(I)$  be the collection of dyadic subintervals of  $I$  with  $|P| = 2^{-k}|I|$ . If  $P \in \mathcal{P}_k(I)$  there holds  $\text{dist}(P, I^{\text{out}}) \geq |I| = 2^k|P|$ . Moreover

$$\sum_{P \in \mathcal{P}_k(I)} |P| = |I|, \quad \inf_{x \in 3P} M_1 h(x) \lesssim 2^k \inf_{x \in 3I} M_1 h(x)$$

for all locally integrable  $h$ . Since

$$\mathbb{T}(I) \subset \bigcup_{k=0}^{\infty} \bigcup_{P \in \mathcal{P}_k(I)} \text{box}(P)$$

we obtain, using the outer Hölder inequality (3.11) to pass to the third line, the chain of inequalities

$$\begin{aligned}
\mathbf{B}_I(f\mathbb{1}_{I^{\text{in}}}, g\mathbb{1}_{I^{\text{out}}}) &\leq \sum_{k \geq 0} \sum_{P \in \mathcal{P}_k(I)} \int_{\text{box}(P)} F(f\mathbb{1}_{I^{\text{in}}}) A(g\mathbb{1}_{I^{\text{out}}}) \, dudtd\eta \\
&\leq \sum_{k \geq 0} \sum_{P \in \mathcal{P}_k(I)} \|F(f\mathbb{1}_{I^{\text{in}}})\mathbb{1}_{\text{box}(P)}\|_{L^\sigma(s^e)} \|A(g\mathbb{1}_{I^{\text{out}}})\mathbb{1}_{\text{box}(P)}\|_{L^\tau(s^m)} \\
&\lesssim \sum_{k \geq 0} \sum_{P \in \mathcal{P}_k(I)} |P| \left( \inf_{x \in 3P} M_p f(x) \right) \left( 2^{-99k} \inf_{x \in 3P} M_p g(x) \right) \\
&\leq \sum_{k \geq 0} 2^{-98k} \sum_{P \in \mathcal{P}_k(I)} |P| \left( \inf_{x \in 3I} M_p f(x) \right) \left( \inf_{x \in 3I} M_1 g(x) \right) \lesssim |I| \left( \inf_{x \in 3I} M_p f(x) \right) \left( \inf_{x \in 3I} M_1 g(x) \right)
\end{aligned}$$

which, by virtue of (3.18), complies with (3.22).

*Proof of Lemma 3.9.* We show how estimate (3.23) follows from Proposition 3.7. Then, (3.24) is obtained from Proposition 3.6 in a similar manner. By quasi-sublinearity and monotonicity of the outer measure  $L^\tau(s^m)$  norm we have that

$$\begin{aligned}
&\|A(h)\mathbb{1}_{\text{box}(P)}\|_{L^\tau(s^m)} \\
&\leq C \|A(h\mathbb{1}_{9P})\mathbb{1}_{\text{box}(P)}\|_{L^\tau(s^m)} + \sum_{k=3}^{\infty} C^k \|A(h\mathbb{1}_{3^k P \setminus 3^{k-1} P})\mathbb{1}_{\text{box}(P)}\|_{L^\tau(s^m)}. \tag{3.25}
\end{aligned}$$

Applying the embedding bound (3.14) with  $c = 3^{-2}$  and  $Q = 3P$  provides us with  $V_{h,1,3P}$  such that  $\text{box}(P) \subset \mathbb{T}(9P) \setminus \mathbb{T}(V_{h,1,3P})$ , whence

$$\|A(h\mathbb{1}_{9P})\mathbb{1}_{\text{box}(P)}\|_{L^\tau(s^m)} \leq CK |P|^{\frac{1}{\tau}} \langle h \rangle_{9P,1} \leq CK |P|^{\frac{1}{\tau}} \inf_{x \in 3P} M_1 h(x).$$

Indeed, we chose  $c$  in such a way that  $|V_{h,1,3P}| < 3^{-1}Q$ , which guarantees that  $\mathbb{T}(V_{h,1,9P})$  does not intersect  $\text{box}(P)$ . We claim that similarly we have that for  $k > 2$  and for an arbitrarily large  $N \gg 1$  there holds

$$\|A(h\mathbb{1}_{3^k P \setminus 3^{k-1} P})\mathbb{1}_{\text{box}(P)}\|_{L^\tau(s^m)} \leq CK 3^{-Nk} |P|^{\frac{1}{\tau}} \langle h \rangle_{3^k P,1} \leq CK |P|^{\frac{1}{\tau}} 3^{-Nk} \inf_{x \in 3P} M_1 h(x).$$

Let

$$(u, t, \eta) \mapsto \Psi_{t,\eta}^{\xi_-, \xi_+}(\cdot - u)$$

be a choice of truncated wave packets which approximately achieves the supremum in

$$A(h\mathbb{1}_{3^k P \setminus 3^{k-1} P})(u, t, \eta),$$

cf. (3.6). Then

$$\tilde{\Psi}_{t,\eta}^{\xi_-, \xi_+}(\cdot - u) := \left( 1 + \frac{|(x-u)-c(P)|}{|P|} \right)^{2N} \Psi_{t,\eta}^{\xi_-, \xi_+}(\cdot - u)$$

are adapted truncated wave packets as well since multiplying by a polynomial does not change the frequency support of  $\Psi_{t,\eta}^{\xi_-, \xi_+}$  and so the conditions on being truncated wave packets is maintained.

Let  $\tilde{A}(h\mathbb{1}_{3^k P \setminus 3^{k-1} P})(u, t, \eta)$  be the embedding obtained by using the wave packets  $\tilde{\Psi}_{t,\eta}^{\xi_-, \xi_+}(\cdot - u)$  instead of  $\Psi_{t,\eta}^{\xi_-, \xi_+}(\cdot - u)$ . Given that  $(u, t, \eta) \in \text{box}(P)$  we have that

$$|A(h\mathbb{1}_{3^k P \setminus 3^{k-1} P})(u, t, \eta)| \leq C 3^{-2Nk} \tilde{A}(h\mathbb{1}_{3^k P \setminus 3^{k-1} P})(u, t, \eta).$$

However the bounds (3.14) also hold for  $\tilde{A}$  with an additional multiplicative constant that depends at most on  $N$ . Applying these bounds with  $P = 3^{k-1}Q$  and  $c = 3^{-k}$  we have once again that

$$\|\tilde{A}(h\mathbb{1}_{3^k P \setminus 3^{k-1} P})\mathbb{1}_{\text{box}(P)}\|_{L^r(s^m)} \leq CK|P|^{\frac{1}{r}}3^k \langle h \rangle_{3^k P, 1}.$$

As long as  $N$  is chosen large enough with respect to  $C > 1$  appearing in (3.25), the above display gives the required bound. The decay factor in term of  $\text{dist}(P, \text{supp } h)$  follows from the fact that the the first  $k_0$  terms in (3.25) vanish if  $\text{supp } h \cap 3^{k_0}P = \emptyset$ .  $\square$

### 3.4.2 The iteration argument

With Lemma 3.8 in hand, we proceed to the proof of Proposition 3.4. Fix  $f, g \in L^\infty(\mathbb{R})$  with compact support. By an application of Fatou's lemma, it suffices to prove (3.8) with  $B_{Q_0}$  in lieu of  $B$  for an arbitrary interval  $Q_0$  with  $\text{supp } f, \text{supp } g \subset Q_0$ . That is, it suffices to construct a sparse collection  $\mathcal{S}$  such that

$$B_{Q_0}(f, g) \leq C \sum_{I \in \mathcal{S}} |I| \langle f \rangle_{I, p} \langle g \rangle_{I, 1} \quad (3.26)$$

provided that the constant  $C$  does not depend on  $Q_0$ . We fix such a  $Q_0$ . Furthermore, as

$$B_{Q_0}(f, g) = \sup_{\varepsilon > 0} B_{Q_0, \varepsilon}(f, g), \quad B_{Q, \varepsilon}(f, g) := \int_{\mathbb{T}(Q)} F(f)(u, t, \eta) A(g)(u, t, \eta) \mathbb{1}_{\{t > \varepsilon\}} \, dudt d\eta$$

it suffices to prove (3.26) with  $B_{Q_0, \varepsilon}$  replacing  $B_{Q_0}$ , with constants uniform in  $\varepsilon > 0$ . We also notice that Lemma 3.8 holds uniformly, if one replaces all instances of  $B_Q$  in (3.15) by  $B_{Q, \varepsilon}$ . From here onwards we fix  $\varepsilon > 0$  and drop it from the notation.

We now perform the following iterative procedure. Set  $\mathcal{S}_0 = \{Q_0\}$ . Suppose that the collection of open intervals  $Q \in \mathcal{S}_n$  has been constructed, and define inductively

$$\mathcal{S}_{n+1} = \bigcup_{Q \in \mathcal{S}_n} \mathcal{I}_Q$$

where  $\mathcal{I}_Q$  is obtained as in the Lemma 3.8. It can be seen inductively that

$$Q \in \mathcal{S}_n \implies |Q| \leq 2^{-12n} |Q_0|.$$

We iterate this procedure as long as  $n \leq N$ , where  $N$  is taken so that  $2^{-12N} |Q_0| < \varepsilon$  holds. At that point we stop the iteration and set

$$\mathcal{S}^* = \bigcup_{n=0}^N \mathcal{S}_n.$$

Making use of estimate (3.17) along the iteration of Lemma 3.8 we readily obtain

$$B_{Q_0}(f, g) \lesssim \sum_{n=0}^{N-1} \sum_{Q \in \mathcal{S}_n} |Q| \langle f \rangle_{3Q, p} \langle g \rangle_{3Q, 1} + \sum_{Q \in \mathcal{S}_N} \sum_{I \in \mathcal{I}_Q} B_I(f\mathbb{1}_{3I}, g\mathbb{1}_{3I}) = \sum_{Q \in \mathcal{S}^*} |Q| \langle f \rangle_{3Q, p} \langle g \rangle_{3Q, 1}$$

as each term  $B_I$ ,  $I \in \mathcal{S}_N$  vanishes by the condition on  $N$ . Now, observing that the sets

$$X_Q := Q \setminus \left( \bigcup_{I \in \mathcal{S}^*: I \subsetneq Q} I \right) = Q \setminus \left( \bigcup_{I \in \mathcal{I}_Q} I \right) \quad Q \in \mathcal{S}^*$$

are pairwise disjoint and, from (3.16),  $|Q \setminus X_Q| \geq (1 - 2^{-12})|Q|$  yields that  $\mathcal{S}^*$  is sparse, and so is  $\mathcal{S} = \{3Q : Q \in \mathcal{S}^*\}$ . This completes the proof of Proposition 3.4.

# Acknowledgements

I would like to thank my advisor, Prof. Dr. Christoph Thiele, for his invaluable guidance and for making this work possible. He has always been very generous with the time he dedicated to me, listening to my ideas and suggesting new ones. Furthermore, his method of discussion, very maieutic in nature, was crucial for bringing out new and interesting ideas, clearing up misconceptions, but at the same helped to build up my confidence in tackling challenging problems. Prof. Thiele managed to be an excellent advisor also on a personal level, striking what I feel is a perfect balance between giving me suggestions and leaving space for independent inquiry. His advice on personal issues and on how to navigate the environment of academia was a priceless gift for a person who aims to be part of it some day. For all this, I am very grateful.

I would also like to thank my Master's advisor, Prof. Fulvio Ricci, who made me discover the wonderful field of mathematics that I work in. He has always been very supportive, even after I came to Bonn.

Finally, this thesis would not have been possible without Prof. Francesco Di Plinio. His friendly demeanor, an intense passion for the subject, and the trust that he accorded me makes him an extraordinary colleague and mentor.

I would need to write another thesis to be able to fully express my due gratitude to all the people who have accompanied me in this endeavor. I was lucky that the people around me made it possible for me to wake up every morning with a smile, looking forward to come to work to learn about mathematics and to work in an environment in which I felt valued, but also to which I felt that I could contribute with my presence.

I also wish to thank my family. I am ever in their debt for laying the foundations that I rely on every day.



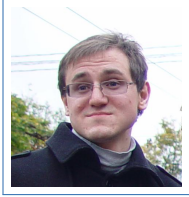


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# Curriculum Vitae



# Gennady Uraltsev

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## Curriculum Vitae

### Occupational field **Mathematics/Accademia**

**Current Position** PhD Student in Mathematics, University of Bonn

Supervisor Prof. Dr. Christoph Thiele

Area of Research Harmonic Analysis, Time-Frequency Analysis

Affiliation BIGS - Math Institute, University of Bonn, Germany

Starting date Oct, 2013

Expected end July, 2017

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### Publications

1. G. Uraltsev "Variational Carleson Embeddings Into The Upper 3-Space". *arXiv:1610.07657*, submitted.
2. F. Di Plinio, Y. Do, G. Uraltsev "Positive Sparse Domination Of Variational Carleson Operators". *arXiv:1612.03028*, submitted preprint.
3. C. Mantegazza, G. Mascellani, and G. Uraltsev "On the distributional Hessian of the distance function". *Pacific Journal of Mathematics* 270.1 (2014): 151-166.

#### Master thesis

title *Multi-parameter Singular Integrals: Product and Flag Kernels*

supervisor Prof. Fulvio Ricci

Institution Pisa State University/Scuola Normale Superiore

#### Bachelor thesis

title *Regularity of Minimizers of One-Dimensional Scalar Variational Problems with Lagrangians with Reduced Smoothness Conditions*

supervisor Prof. Luigi Ambrosio

Institution Pisa State University/Scuola Normale Superiore

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### Education

*Math Institute, Uni-Bonn – Endenicher Allee, 60 – 53115, Bonn – Germany*

✉ [guraltse@math.uni-bonn.de](mailto:guraltse@math.uni-bonn.de)

🌐 <http://www.math.uni-bonn.de/people/guraltse/> • Nationality: *Russia*

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- 2011–2013 **Master of Science in Mathematics**, *Department of Pure and Applied Mathematics, University of Pisa, Pisa, Italy, 110/110 Cum Laude.*
- 2011–2013 **Diploma of Mathematics (Class of Science)**, *Scuola Normale Superiore, Pisa, Italy, 70/70.*
- 2008–2011 **Bachelor of Science in Mathematics**, *Department of Pure and Applied Mathematics, University of Pisa, Pisa, Italy, 110/110 Cum Laude.*
- 2008–2011 **First level Diploma of Mathematics (Class of Science)**, *Scuola Normale Superiore, Pisa, Italy.*

## Talks and Seminars

### Talks

- December 2016 **St. Petersburg Chebychev laboratory minicourse** *Invited speaker: “Time-frequency analysis of modulation invariant operators using outer measure spaces”*
- Nov.-Dec. 2016 **St. Petersburg Department of Steklov Institute of Mathematics of Russian Academy of Sciences** *Invited speaker*  
**Math Department, Brown University** *Invited speaker*  
**Math Department, Michigan State University** *Invited speaker*  
**Math Department, Cornell University** *Invited speaker*  
 “Variational Carleson and beyond using embedding maps and iterated outer measure spaces”
- Feb.-Mar. 2016 **Math Department, Yale University** *Invited speaker*  
**Math Department, Brown University** *Invited speaker*  
 “Time frequency analysis below local  $L^2$ : Iterated outer measure  $L^p$  spaces”
- 2013-2016 **Math Department, Bonn University** *speaker at research group seminar (various topics)*

### Seminars and Summer Schools

- Sept, 2016 Harmonic Analysis and Rough Paths *speaker* “A signed measure on rough paths associated to a PDE of high order: results and conjectures.” – after D Levin, and T Lyons
- Sept, 2015 Sharp Inequalities in Harmonic Analysis *speaker* “A mass-transportation approach to sharp Sobolev and Gagliardo–Nirenberg inequalities.” – after Cordero-Erausquin, Nazaret, and Villani.
- May, 2015 Blow-up for non-linear Dispersive PDEs *speaker* “On the blow up phenomenon for the  $L^2$  critical nonlinear Schrödinger Equation” – after Raffaël.
- May, 2014 Carleson theorems and Radon type behavior *speaker* “The (weak- $L^2$ ) boundedness of the quadratic Carleson operator” – after Lie.

## Teaching experience

- Oct 2014 - Introduction to Real and Harmonic Analysis *Teaching Assistant*, Universität Bonn,  
 Feb 2015 Professor: Prof. Dr. Herbert Koch

*Math Institute, Uni-Bonn – Endenicher Allee, 60 – 53115, Bonn – Germany*

✉ [guraltse@math.uni-bonn.de](mailto:guraltse@math.uni-bonn.de)

🌐 <http://www.math.uni-bonn.de/people/guraltse/> • Nationality: *Russia*

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March 2016 - Complex Analysis *Course notes*, Universität Bonn, Professor: Prof. Dr. Christoph Thiele  
Sept 2016  
March 2015 - Analysis 2 *Course notes*, Universität Bonn, Professor: Prof. Dr. Christoph Thiele  
Sept 2015

## Languages

Russian C2 - Proficient– native language  
Italian C2 - Proficient  
English C2 - Proficient  
French B2 - Independent user  
German A2 -Basic user

### Certificates

2005 Certificate of Advanced English, University of Cambridge ESOL *Grade: A*  
2012 TOEFL iBT *Reading: 30/30, Listening: 30/30, Speaking: 28/30, Writing: 30/30, Total: 118/120*

## Computer skills

**Administration of a computer cluster/ network administration** level: advanced.

**Using Linux** level: advanced.

**Using document markup software ( $\LaTeX$ , Office Suite Document Writing/Spreadsheet/Presentation):**  
level: medium.

## Other

### Awards and Scholarships

- o Scuola Normale Superiore: Full Scholarship Award winner for a 5-year term of studies in the Science Class of Scuola Normale Superiore di Pisa. Total scholarships: 30 per year

### Olympiads

- o International Physics Olympiad 2008, Vietnam : member of the Italian team, Honourable Mention
- o National Physics Olympiad 2008, Italy: Gold Medal
- o National Mathematics Olympiad 2008, Italy: Gold Medal
- o National Physics Olympiad 2007, Italy: Gold Medal

## References

Name Prof. Dr. Christoph Thiele  
Position / Hausdorff Chair (W3), Bonn (Germany)  
Organization  
Email thiele@math.uni-bonn.de

Name Prof. Dr. Herbert Koch  
Position / Full Professor (W3), Bonn (Germany)  
Organization

*Math Institute, Uni-Bonn – Endenicher Allee, 60 – 53115, Bonn – Germany*

✉ [guraltse@math.uni-bonn.de](mailto:guraltse@math.uni-bonn.de)

🌐 <http://www.math.uni-bonn.de/people/guraltse/> • Nationality: *Russia*

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Email koch@math.uni-bonn.de

Name Prof. Dr. Alexander Volberg

Position / University Distinguished Professor, Michigan State University  
Organization

Email volberg@math.msu.edu

Name Prof. Dr. Massimiliano Gubinelli

Position / Hausdorff Chair (W3), Bonn (Germany)  
Organization

Email gubinelli@math.uni-bonn.de

Name Prof. Dr. Fulvio Ricci

Position / Full Professor, Classe di Scienze, Scuola Normale Superiore di Pisa (Italy)  
Organization

Email fricci@sns.it