

Group actions and Burnside's lemma

Combinatorics

Fall 2026

A necklace is a ring of colored beads, and two necklaces are the same if one can be rotated into the other. How many necklaces have 4 beads, each black or white? Recording the colors at positions 1, 2, 3, 4 counterclockwise gives $2^4 = 16$ strings, and [Figure 14.1](#) sorts them into classes of strings that are rotations of one another. There are 6 classes. Dividing 16 by the 4 rotations gives the wrong answer 4, because the classes have different sizes: **BWBW** is unchanged by a half turn and has only 2 rotations, and **BBBB** has only 1. This note develops the language of group actions, explains the class sizes with the orbit–stabilizer theorem, and counts classes with Burnside's lemma.

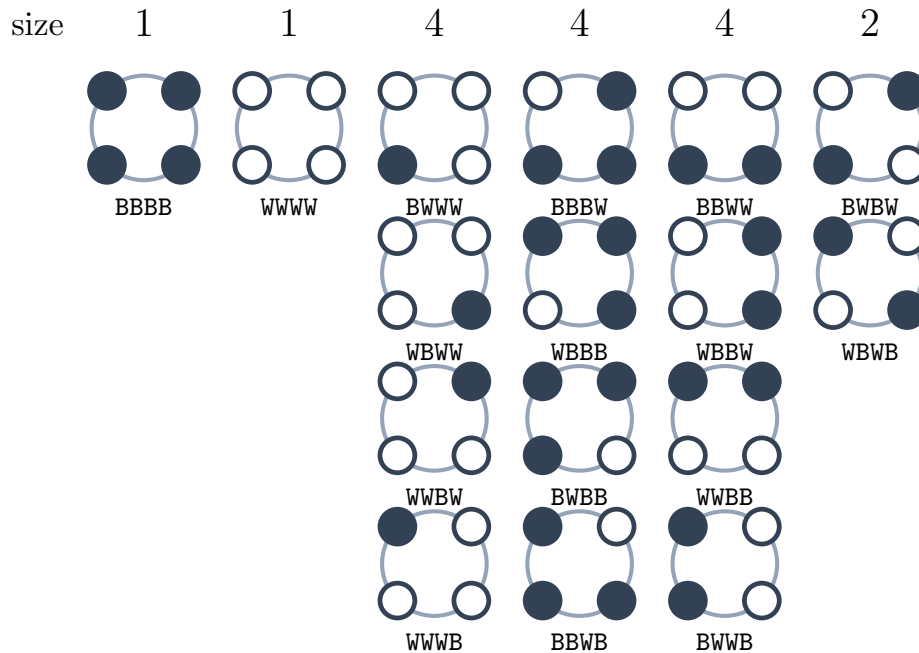


Figure 14.1: The 16 black-and-white colorings of four positions, sorted into classes of colorings that are rotations of one another. The number above each column is the size of the class. Position 1 is at the bottom left, and the positions continue counterclockwise.

Figure description: Six columns of four-bead necklaces: all black (1); all white (1); one black bead (4); three black beads (4); two adjacent black beads (4); two opposite black beads (2). Each necklace is labeled by its string, such as **BBWW**.

After studying these notes, you will be able to do the following.

- Recognize a group action, including the action of a group on colorings.
- Compute orbits and stabilizers, and use the orbit–stabilizer theorem to find orbit sizes.
- State and prove Burnside's lemma.
- Count the colorings fixed by a group element from its number of cycles.
- Count strings, necklaces, bracelets, grid colorings, and cube colorings up to symmetry.

Throughout, groups and their notation are as in [Lecture Note 14](#).

14.1 Group actions

Definition 14.1 (Action). An **action** of a group G on a finite set Y assigns to every $g \in G$ and $y \in Y$ an element $g \cdot y \in Y$ such that $e \cdot y = y$ and $g \cdot (h \cdot y) = (gh) \cdot y$ for all $g, h \in G$ and $y \in Y$.

For each g , the function $y \mapsto g \cdot y$ is a permutation of Y , since $y \mapsto g^{-1} \cdot y$ undoes it: $g^{-1} \cdot (g \cdot y) = (g^{-1}g) \cdot y = y$. So an action lets each group element rearrange Y , compatibly with the group operation.

Example 14.2 (Actions). Each of the following is an action.

- A permutation group G on X acts on X by $g \cdot x = g(x)$. For instance, D_4 acts on the corners of the square.
- A group G acts on itself by $g \cdot x = gx$; these are the permutations λ_g of Cayley's theorem.
- A subgroup H of G acts on G by $h \cdot x = xh^{-1}$. The rules hold because $h \cdot (k \cdot x) = xk^{-1}h^{-1} = x(hk)^{-1} = (hk) \cdot x$.
- S_n acts on the 2-element subsets of $\{1, \dots, n\}$ by $g \cdot \{a, b\} = \{g(a), g(b)\}$.

The action we need for counting moves colorings. Let X be a finite set of positions and K a set of k colors. A **coloring** is a function $c: X \rightarrow K$; there are $k^{|X|}$ colorings. When $X = \{1, \dots, n\}$ we write c as the string $c(1)c(2)\dots c(n)$. If G acts on X , moving a coloring by g carries the color at each position x to the position $g \cdot x$. So $g \cdot c$ is the coloring with $(g \cdot c)(g \cdot x) = c(x)$, or equivalently

$$(g \cdot c)(x) = c(g^{-1} \cdot x) \quad \text{for every } x \in X. \quad (14.1)$$

For example, $r = (1\ 2\ 3\ 4)$ moves the colors of BBWW one position counterclockwise: $r \cdot \text{BBWW} = \text{WBBW}$.

Proposition 14.3 (Moving colorings is an action). Formula (14.1) defines an action of G on the colorings of X .

Proof. First, $(e \cdot c)(x) = c(e \cdot x) = c(x)$. Second, using $(gh)^{-1} = h^{-1}g^{-1}$,

$$(g \cdot (h \cdot c))(x) = (h \cdot c)(g^{-1} \cdot x) = c(h^{-1} \cdot (g^{-1} \cdot x)) = c((gh)^{-1} \cdot x) = ((gh) \cdot c)(x).$$

□

14.2 Orbits and stabilizers

Definition 14.4 (Orbit, stabilizer, fixed set). Let G act on Y .

The **orbit** of y is $G \cdot y = \{g \cdot y : g \in G\}$.

The **stabilizer** of y is $\text{Stab}(y) = \{g \in G : g \cdot y = y\}$.

The **fixed set** of g is $\text{Fix}(g) = \{y \in Y : g \cdot y = y\}$.

The columns of Figure 14.1 are the orbits of the rotation group C_4 on the 16 colorings. A **necklace** with n beads and k colors is an orbit of the colorings of the corners of a regular n -gon under the rotation group C_n , and a **bracelet** is an orbit under the full symmetry group D_n , which also allows turning the ring over.

Proposition 14.5 (Orbits and stabilizers). Every element of Y lies in exactly one orbit, and every stabilizer is a subgroup of G .

Proof. Since $e \cdot y = y$, y lies in its own orbit. If $z \in G \cdot y$, say $z = g \cdot y$, then $h \cdot z = (hg) \cdot y$ and $h \cdot y = (hg^{-1}) \cdot z$, so $G \cdot z = G \cdot y$; hence two orbits with a common element z are both equal to $G \cdot z$. For the stabilizer, $e \cdot y = y$, and if $g \cdot y = h \cdot y = y$ then $(gh) \cdot y = g \cdot (h \cdot y) = y$. By the subgroup test for finite sets in [Lecture Note 14](#), a nonempty finite subset closed under the operation is a subgroup. $\square$

Under C_4 , the stabilizer of BWBW is $\{e, r^2\}$, and its orbit has 2 elements; the stabilizer of BBWW is $\{e\}$, and its orbit has 4; the stabilizer of BBBB is C_4 , and its orbit has 1. In each case the product of the two numbers is $4 = |C_4|$.

Theorem 14.6 (Orbit–stabilizer theorem). Let a finite group G act on Y and let $y \in Y$. Then

$$|G \cdot y| \cdot |\text{Stab}(y)| = |G|.$$

Proof. For $z \in G \cdot y$, let $A_z = \{g \in G : g \cdot y = z\}$. Every $g \in G$ lies in exactly one set A_z , namely the one with $z = g \cdot y$, so by the sum rule $|G| = \sum_{z \in G \cdot y} |A_z|$. We show that each A_z has $|\text{Stab}(y)|$ elements. Fix $g \in A_z$. If $\tau \in \text{Stab}(y)$, then $(g\tau) \cdot y = g \cdot y = z$, so $g\tau \in A_z$. Conversely, if $g' \in A_z$, then $(g^{-1}g') \cdot y = g^{-1} \cdot z = y$, so $g^{-1}g' \in \text{Stab}(y)$. The maps $\tau \mapsto g\tau$ and $g' \mapsto g^{-1}g'$ undo each other, so they are inverse bijections between $\text{Stab}(y)$ and A_z . Hence $|G| = |G \cdot y| \cdot |\text{Stab}(y)|$. $\square$

Example 14.7 (Orders of symmetry groups). The symmetries D_n of a regular n -gon act on its corners. A rotation moves corner 1 to any corner, so the orbit of corner 1 has n elements. A symmetry fixing corner 1 must send corner 2 to one of the two corners adjacent to 1, and those two choices determine the symmetry, as for the square; so the stabilizer has 2 elements, and $|D_n| = 2n$.

The rotations of a cube act on its 6 faces. Any face can be rotated to any other, and the rotations that keep a given face in place are the 4 rotations about the axis through its center. So the cube has $6 \cdot 4 = 24$ rotations.

The orbit–stabilizer theorem also proves Lagrange’s theorem, stated in [Lecture Note 14](#).

Corollary 14.8 (Lagrange’s theorem). If H is a subgroup of a finite group G , then $|H|$ divides $|G|$.

Proof. Let H act on G by $h \cdot x = xh^{-1}$, as in [Example 14.2](#). If $h \cdot x = x$, then $xh^{-1} = x$, and cancellation gives $h = e$; so every stabilizer is $\{e\}$, and by the orbit–stabilizer theorem every orbit has exactly $|H|$ elements. The orbits split the $|G|$ elements of G into blocks of $|H|$ elements each, so $|H|$ divides $|G|$. $\square$

14.3 Burnside’s lemma

Theorem 14.9 (Burnside’s lemma). Let a finite group G act on a finite set Y . The number of orbits is

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.$$

Proof. We count the set F of pairs (g, y) with $g \cdot y = y$ in two ways. For each g there are $|\text{Fix}(g)|$ such pairs, so $|F| = \sum_g |\text{Fix}(g)|$. For each y there are $|\text{Stab}(y)|$ such pairs, and by the orbit–stabilizer theorem $|\text{Stab}(y)| = |G|/|G \cdot y|$. If O is an orbit, its elements contribute

$$\sum_{y \in O} \frac{|G|}{|O|} = |G|$$

pairs. So if there are N orbits, $|F| = N|G|$, and comparing the two counts gives the formula. $\square$

The number of orbits is the *average* number of elements fixed by a group element. Large orbits have small stabilizers and small orbits have large stabilizers, and the double count gives every orbit the same weight $|G|$.

For colorings, the fixed sets are determined by cycles. Write $\text{cyc}(g)$ for the number of cycles, including fixed points, of the permutation $x \mapsto g \cdot x$ of the positions.

Theorem 14.10 (Fixed colorings). Let G act on a finite set X of positions, and let there be k colors. A coloring c satisfies $g \cdot c = c$ if and only if c is constant on each cycle of g . Hence $|\text{Fix}(g)| = k^{\text{cyc}(g)}$.

Proof. By (14.1), $g \cdot c = c$ means $c(g^{-1} \cdot x) = c(x)$ for all x ; replacing x by $g \cdot x$, this says $c(x) = c(g \cdot x)$ for all x . That holds exactly when consecutive positions $x, g \cdot x, g^2 \cdot x, \dots$ of every cycle have the same color, that is, when c is constant on each cycle. Choosing one color per cycle gives $k^{\text{cyc}(g)}$ fixed colorings. $\square$

Warning 14.11 (Fixed points are cycles). Cycle notation omits fixed points, but $\text{cyc}(g)$ counts them. For example, $d_{13} = (2\ 4)$ has $\text{cyc}(d_{13}) = 3$ on the corners of the square and fixes k^3 colorings, not k .

Combining the two theorems, the number of orbits of colorings with k colors is

$$\frac{1}{|G|} \sum_{g \in G} k^{\text{cyc}(g)}.$$

In each example below, the work is to list the elements of G and count their cycles on the positions.

14.4 Examples

14.4.1 Strings up to reversal

Reading a string of length n backwards gives its reversal. How many strings of length n over k letters are there, if a string and its reversal count as the same? The group is $\{e, \rho\}$, where ρ sends position i to position $n + 1 - i$. The cycles of ρ are the pairs $\{i, n + 1 - i\}$, together with the middle position when n is odd, so $\text{cyc}(\rho) = \lceil n/2 \rceil$. The answer is

$$\frac{k^n + k^{\lceil n/2 \rceil}}{2}.$$

For $n = 4$ and $k = 2$, this is $(16 + 4)/2 = 10$: the 4 palindromes are fixed by ρ , and the other 12 strings form 6 pairs.

14.4.2 Necklaces

For the opening problem, the rotations of the square have cycle counts $\text{cyc}(e) = 4$, $\text{cyc}(r) = \text{cyc}(r^3) = 1$, and $\text{cyc}(r^2) = 2$, so there are

$$\frac{2^4 + 2^1 + 2^2 + 2^1}{4} = \frac{24}{4} = 6$$

necklaces with 4 beads and 2 colors, as in Figure 14.1.

For n beads, the rotation r^j moves every position j steps. Its cycles are described by the greatest common divisor $\gcd(n, j)$, with the convention $\gcd(n, 0) = n$. We use one fact from elementary number theory: if a and b have no common divisor greater than 1 and a divides mb , then a divides m .

Proposition 14.12 (Cycles of a rotation). For $0 \leq j \leq n-1$ and $d = \gcd(n, j)$, the rotation r^j of n positions has d cycles, each of length n/d .

Proof. Applying r^j to a position m times moves it mj steps, which returns it to its start exactly when n divides mj . So every cycle has length equal to the smallest $m \geq 1$ with $n \mid mj$. For $j = 0$ this is $1 = n/d$. Otherwise write $n = dn'$ and $j = dj'$, where n' and j' have no common divisor greater than 1; then $n \mid mj$ exactly when $n' \mid mj'$, exactly when $n' \mid m$. So the cycles have length $n' = n/d$, and there are $n/(n/d) = d$ of them. $\square$

Corollary 14.13 (Number of necklaces). There are $\frac{1}{n} \sum_{j=0}^{n-1} k^{\gcd(n,j)}$ necklaces with n beads and k colors.

For $n = 6$, the values $\gcd(6, j)$ for $j = 0, \dots, 5$ are 6, 1, 2, 3, 2, 1, and with $k = 2$ there are $(64 + 2 + 4 + 8 + 4 + 2)/6 = 14$ necklaces. If $n = p$ is prime, every rotation other than e has one cycle, and there are $(k^p + (p-1)k)/p$ necklaces. Since this number is an integer, p divides $k^p + (p-1)k - pk = k^p - k$. This is Fermat's little theorem, obtained here by counting necklaces.

14.4.3 Bracelets

For bracelets we also need the reflections. Computing with corner labels modulo n (so that 0 means corner n), the reflections of the n -gon are the permutations

$$s_j(i) = j + 2 - i \pmod{n}, \quad 0 \leq j \leq n-1.$$

Each s_j places the corners in clockwise order, the n maps s_j have different values $s_j(1) = 1 + j$, and together with the n rotations they give all $2n$ symmetries. Each s_j exchanges the corners i and $j + 2 - i$, so its cycles have length 1 or 2, and its fixed points solve $2i \equiv j + 2 \pmod{n}$. For the square, $s_0 = d_{13}$, $s_1 = v$, $s_2 = d_{24}$, $s_3 = h$.

- If n is odd, each s_j has one fixed point, so $\text{cyc}(s_j) = (n+1)/2$.
- If n is even, the $n/2$ reflections with j even have two fixed points, so $\text{cyc} = n/2 + 1$; the $n/2$ with j odd have none, so $\text{cyc} = n/2$.

Adding these terms to the rotation terms, the number of bracelets is

$$\frac{1}{2n} \left(\sum_{j=0}^{n-1} k^{\gcd(n,j)} + R \right), \quad R = \begin{cases} n k^{(n+1)/2}, & n \text{ odd}, \\ \frac{n}{2} (k^{n/2+1} + k^{n/2}), & n \text{ even}. \end{cases}$$

With $n = 6$ and $k = 2$, $R = 3(16 + 8) = 72$, and there are $(84 + 72)/12 = 13$ bracelets, one fewer than necklaces. With $n = 4$ and $k = 3$, there are $(81 + 3 + 9 + 3)/4 = 24$ necklaces and $(96 + 2(27 + 9))/8 = 21$ bracelets.

14.4.4 Black-and-white images on a three-by-three grid

How many black-and-white images on a 3×3 grid of pixels are there, if images that differ by a rotation or reflection of the grid count as the same? The group is D_4 , now acting on the 9 cells.

- e has 9 cycles.

- The quarter turns r and r^3 fix the center and move the four corner cells in one 4-cycle and the four edge cells in another: 3 cycles.
- The half turn r^2 fixes the center and exchanges opposite cells in 4 pairs: 5 cycles.
- Each of the four reflections fixes the 3 cells on its axis and exchanges the other 6 in 3 pairs: 6 cycles.

So the number of images is

$$\frac{2^9 + 2 \cdot 2^3 + 2^5 + 4 \cdot 2^6}{8} = \frac{512 + 16 + 32 + 256}{8} = 102.$$

14.4.5 Colorings of the faces of a cube

By [Example 14.7](#), the cube has 24 rotations. They are sorted by their axes:

- the identity: 6 cycles on the faces;
- rotations by 90° or 270° about an axis through the centers of two opposite faces (3 axes, 6 rotations): the two faces on the axis are fixed and the other four form a 4-cycle, so 3 cycles;
- rotations by 180° about the same axes (3 rotations): two fixed faces and two 2-cycles, so 4 cycles;
- rotations by 120° or 240° about an axis through two opposite vertices (4 axes, 8 rotations): the three faces at each of the two vertices form a 3-cycle, so 2 cycles;
- rotations by 180° about an axis through the midpoints of two opposite edges (6 axes, 6 rotations): the faces are exchanged in 3 pairs, so 3 cycles.

These account for $1 + 6 + 3 + 8 + 6 = 24$ rotations, so the list is complete. With k colors, the number of colorings of the faces up to rotation is

$$\frac{k^6 + 3k^4 + 12k^3 + 8k^2}{24}.$$

For $k = 2$ this is $(64 + 48 + 96 + 32)/24 = 10$, and for $k = 3$ it is 57.

Problem (Check yourself). Count the necklaces and the bracelets with 5 beads and 3 colors. Count the black-and-white images on a 2×2 grid up to rotations and reflections, and check the answer by listing them. Count the colorings of the 8 vertices of a cube with 2 colors up to rotation.