

11. Solving linear recurrences with constant coefficients

Combinatorics

Fall 2026

In the previous note, we saw that many counting problems lead naturally to recurrence relations. For example, the number of tilings of a $2 \times n$ board with 1×2 dominoes satisfies

$$a_n = a_{n-1} + a_{n-2} \quad (n \geq 2),$$

with initial conditions $a_0 = 1$ and $a_1 = 1$. The recurrence allows us to compute subsequent values step by step:

$$a_2 = 2, \quad a_3 = 3, \quad a_4 = 5, \quad a_5 = 8, \quad a_6 = 13, \quad a_7 = 21.$$

This step-by-step calculation works well for small values of n , but it becomes impractical if we need a_{50} or a_{100} . Moreover, generating terms one at a time obscures the qualitative behavior of the sequence: how fast does a_n grow as n becomes large? Is its growth polynomial, exponential, or something else?

We seek a **closed formula**: an explicit expression for a_n as a function of n that can be evaluated directly without computing intermediate terms. For linear recurrences with constant coefficients, the algebraic structure of the recurrence makes this possible. Substituting a geometric sequence $a_n = \lambda^n$ converts the recurrence into an algebraic equation called the **characteristic equation**. The roots of this equation determine the building blocks of all solutions, and the initial conditions select the unique sequence that solves the original problem.

After studying these notes, you will be able to do the following.

- Classify a recurrence by order, linearity, coefficient type, and homogeneity.
- Form the characteristic polynomial of a linear homogeneous recurrence with constant coefficients.
- Solve homogeneous recurrences whose characteristic roots are real, including repeated roots.
- Explain why a root of multiplicity m contributes the terms $r^n, nr^n, \dots, n^{m-1}r^n$.
- Find a particular solution for a driving term $P(n)r^n$, accounting for whether r is a characteristic root.
- Combine a particular solution with the general homogeneous solution before applying initial conditions.
- Express solutions arising from complex roots in terms of a radius, an argument, sines, and cosines.

11.1 Review: classifying recurrence relations

A recurrence relation expresses the general term a_n in terms of earlier terms of the sequence $(a_n)_{n \geq 0}$. We classify recurrences according to four structural features.

- The **order** is the difference between the highest and lowest indices that appear in the equation. For example, the recurrence $a_n = 3a_{n-1} - 5a_{n-4}$ has order 4 because the index difference is $n - (n - 4) = 4$, even though the intermediate terms a_{n-2} and a_{n-3} do not appear.
- The recurrence is **linear** if it can be written in the form

$$c_0(n)a_n + c_1(n)a_{n-1} + \dots + c_d(n)a_{n-d} = g(n),$$

where $c_0(n) \neq 0$ and $c_d(n) \neq 0$. In words, every sequence term appears only to the first power, no sequence terms are multiplied together, and no sequence term appears inside nonlinear functions such as square roots or exponents.

- A linear recurrence has **constant coefficients** if the coefficients $c_0, c_1, \dots, c_d$ are constants that do not depend on n . If any coefficient depends on n , the recurrence has **variable coefficients**.
- A linear recurrence is **homogeneous** if the driving term $g(n)$ is zero for all n , so that every nonzero term involves the sequence. It is **nonhomogeneous** if $g(n) \neq 0$ for at least one n .

The terms *homogeneous* and *nonhomogeneous* describe linear recurrences in this note. A nonlinear recurrence requires different techniques and is not assigned either label here.

Example 1 (Classifying before choosing a method). Consider the following recurrences.

1. The recurrence

$$a_n = 7a_{n-1} - 10a_{n-2}$$

has order 2. It is linear, has constant coefficients, and is homogeneous. The characteristic-root method applies directly.

2. The recurrence

$$b_n = 7b_{n-1} - 10b_{n-2} + (3n - 1)2^n$$

also has order 2 and constant coefficients, but it is nonhomogeneous. Its driving term is $(3n - 1)2^n$. We find one particular solution to account for this driving term, and then add the solutions of the corresponding homogeneous recurrence.

3. The recurrence

$$c_n = nc_{n-1} + c_{n-2}$$

is linear, homogeneous, and of order 2, but it has variable coefficients because the coefficient n changes with the index. The constant-coefficient method in this note does not apply.

4. The recurrence

$$d_n = d_{n-1}d_{n-2} + 1$$

has order 2 and is nonlinear because two earlier sequence terms are multiplied together. It does not possess a characteristic polynomial.

5. The recurrence

$$e_n = 4e_{n-1} + e_{n-3} - 5$$

has order 3, not order 2: the coefficient of e_{n-2} is zero, but the equation still reaches back to e_{n-3} . It is linear, has constant coefficients, and is nonhomogeneous with constant driving term -5 .

We consider finite-order linear recurrences with constant coefficients, beginning with the homogeneous case where every term involves the sequence and the driving term is zero.

11.2 Homogeneous recurrences with real characteristic roots

We consider a linear homogeneous recurrence of order d with constant coefficients, written in standard form as

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \dots + c_da_{n-d} \quad (n \geq d), \tag{1}$$

where $c_1, c_2, \dots, c_d$ are real constants and $c_d \neq 0$. Moving all terms to the left-hand side gives

$$a_n - c_1a_{n-1} - c_2a_{n-2} - \dots - c_da_{n-d} = 0.$$

A first-order recurrence $a_n = ca_{n-1}$ has geometric sequences as solutions: $a_n = a_0c^n$. Geometric sequences λ^n have the property that shifting the index multiplies the term by a constant power: $\lambda^{n-k} = \lambda^{-k}\lambda^n$. Let us test whether a geometric sequence $a_n = \lambda^n$ can solve the higher-order recurrence (1). Substituting $a_n = \lambda^n$ into the recurrence gives

$$\lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} - \dots - c_d\lambda^{n-d} = 0.$$

Because we seek a nonzero solution, we require $\lambda \neq 0$. Factoring out λ^{n-d} yields

$$\lambda^{n-d}(\lambda^d - c_1\lambda^{d-1} - c_2\lambda^{d-2} - \dots - c_d) = 0.$$

Since $\lambda \neq 0$, the sequence $a_n = \lambda^n$ satisfies the recurrence if and only if λ is a root of the polynomial in parentheses.

Definition 2 (Characteristic polynomial). The **characteristic polynomial** of the recurrence (1) is

$$p(t) = t^d - c_1t^{d-1} - c_2t^{d-2} - \dots - c_d.$$

The equation $p(t) = 0$ is the **characteristic equation**, and its roots are the **characteristic roots** of the recurrence.

Notice the signs in $p(t)$: each coefficient c_j from the right-hand side of (1) enters $p(t)$ with a minus sign, $-c_j$. Moving all sequence terms to one side of the recurrence before writing down the polynomial prevents sign errors.

11.2.1 Distinct real roots and the opening problem

Suppose the characteristic polynomial $p(t)$ has d distinct real roots $r_1, r_2, \dots, r_d$. Then each of the sequences $(r_1^n)_{n \geq 0}, (r_2^n)_{n \geq 0}, \dots, (r_d^n)_{n \geq 0}$ satisfies the recurrence.

Because the recurrence is linear and homogeneous, the principle of superposition applies: any linear combination of solutions is also a solution. If $a_n^{(1)}$ and $a_n^{(2)}$ both satisfy the recurrence, then for any constants A and B , the sequence $Aa_n^{(1)} + Ba_n^{(2)}$ satisfies it as well:

$$\begin{aligned} \sum_{j=1}^d c_j(Aa_{n-j}^{(1)} + Ba_{n-j}^{(2)}) &= A \sum_{j=1}^d c_j a_{n-j}^{(1)} + B \sum_{j=1}^d c_j a_{n-j}^{(2)} \\ &= Aa_n^{(1)} + Ba_n^{(2)}. \end{aligned}$$

Therefore,

$$a_n = A_1r_1^n + A_2r_2^n + \dots + A_dr_d^n \tag{2}$$

is a solution for every choice of constants $A_1, A_2, \dots, A_d$.

We must also confirm that (2) accounts for all solutions to the recurrence. An order- d recurrence determines a unique sequence once d initial values $a_0, a_1, \dots, a_{d-1}$ are specified. Setting $n = 0, 1, \dots, d-1$ in (2) produces a system of d linear equations in the d variables $A_1, A_2, \dots, A_d$:

$$\begin{aligned} A_1 + A_2 + \dots + A_d &= a_0 \\ A_1r_1 + A_2r_2 + \dots + A_dr_d &= a_1 \\ A_1r_1^2 + A_2r_2^2 + \dots + A_dr_d^2 &= a_2 \\ &\vdots \\ A_1r_1^{d-1} + A_2r_2^{d-1} + \dots + A_dr_d^{d-1} &= a_{d-1}. \end{aligned}$$

Because the roots $r_1, r_2, \dots, r_d$ are distinct, this linear system always has a unique solution. (Writing this system as a matrix-vector equation yields a coefficient matrix known as a Vandermonde matrix. Its

determinant is the product of all differences $(r_j - r_i)$ for $i < j$, which is nonzero precisely when the roots are distinct. Finding the initial data this way is indeed solving a matrix-vector equation, but the details of this matrix determinant are beyond the scope of this course.) Since every sequence satisfying the recurrence is uniquely determined by its initial values, the family (2) contains all solutions.

Example 3 (Resolving the opening problem: closed formula for domino tilings). Find a closed-form formula for the number of tilings a_n of a $2 \times n$ board with 1×2 dominoes, which satisfies

$$a_n = a_{n-1} + a_{n-2} \quad (n \geq 2)$$

with initial conditions $a_0 = 1$ and $a_1 = 1$.

Solution: Moving all terms to the left gives $a_n - a_{n-1} - a_{n-2} = 0$. The characteristic equation is

$$t^2 - t - 1 = 0.$$

By the quadratic formula, the characteristic roots are

$$r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2}.$$

These roots are real and distinct. The general solution is therefore

$$a_n = A \left(\frac{1 + \sqrt{5}}{2} \right)^n + B \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

We use the initial conditions $a_0 = 1$ and $a_1 = 1$ to determine A and B :

$$\begin{aligned} n = 0 : \quad & A + B = 1, \\ n = 1 : \quad & A \left(\frac{1 + \sqrt{5}}{2} \right) + B \left(\frac{1 - \sqrt{5}}{2} \right) = 1. \end{aligned}$$

Multiplying the second equation by 2 gives

$$(A + B) + \sqrt{5}(A - B) = 2.$$

Substituting $A + B = 1$ yields $1 + \sqrt{5}(A - B) = 2$, so

$$A - B = \frac{1}{\sqrt{5}}.$$

Adding and subtracting the equations $A + B = 1$ and $A - B = 1/\sqrt{5}$ gives

$$\begin{aligned} A &= \frac{1 + \frac{1}{\sqrt{5}}}{2} = \frac{\sqrt{5} + 1}{2\sqrt{5}} = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right), \\ B &= \frac{1 - \frac{1}{\sqrt{5}}}{2} = \frac{\sqrt{5} - 1}{2\sqrt{5}} = -\frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right). \end{aligned}$$

Substituting A and B back into the general solution gives the closed formula

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1}.$$

This formula is known as Binet's formula (shifted by one index because $a_0 = 1 = F_1$). Two observations clarify how it behaves:

1. Although $\sqrt{5}$ appears in both terms, every value a_n is an integer. For instance, at $n = 2$,

$$a_2 = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^3 - \left(\frac{1 - \sqrt{5}}{2} \right)^3 \right] = \frac{1}{\sqrt{5}} \left[\frac{16 + 8\sqrt{5}}{8} - \frac{16 - 8\sqrt{5}}{8} \right] = \frac{1}{\sqrt{5}} (2\sqrt{5}) = 2,$$

which matches $a_2 = a_1 + a_0 = 1 + 1 = 2$.

2. Because $|(1 - \sqrt{5})/2| \approx 0.618 < 1$, the second term decays exponentially to zero as n increases. For large n , a_n is the nearest integer to $\frac{1}{\sqrt{5}}((1 + \sqrt{5})/2)^{n+1}$, showing that the number of tilings grows exponentially at the golden ratio rate $\phi \approx 1.618$.

Example 4 (Three distinct real roots). Solve

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \quad (n \geq 3)$$

with initial conditions

$$a_0 = 2, \quad a_1 = 3, \quad a_2 = 7.$$

Solution: Moving all terms to the left gives

$$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0,$$

so the characteristic polynomial is

$$p(t) = t^3 - 6t^2 + 11t - 6 = (t - 1)(t - 2)(t - 3).$$

The three roots 1, 2, 3 are distinct. The general solution therefore has one unknown coefficient for each root:

$$a_n = A \cdot 1^n + B \cdot 2^n + C \cdot 3^n = A + B2^n + C3^n.$$

We now fit the three initial values:

$$\begin{aligned} n = 0 : \quad & A + B + C = 2, \\ n = 1 : \quad & A + 2B + 3C = 3, \\ n = 2 : \quad & A + 4B + 9C = 7. \end{aligned}$$

Subtracting the first equation from the second eliminates A :

$$B + 2C = 1.$$

Subtracting the second equation from the third also eliminates A :

$$2B + 6C = 4, \quad \text{or} \quad B + 3C = 2.$$

Subtracting $B + 2C = 1$ from $B + 3C = 2$ gives $C = 1$. Then $B = 1 - 2(1) = -1$, and finally $A = 2 - B - C = 2 - (-1) - 1 = 2$. Thus

$$\boxed{a_n = 2 - 2^n + 3^n.}$$

As an arithmetic check, the formula predicts $a_3 = 2 - 8 + 27 = 21$, while the recurrence gives

$$a_3 = 6(7) - 11(3) + 6(2) = 42 - 33 + 12 = 21.$$

11.2.2 Repeated real roots and the shift operator

When the characteristic polynomial has repeated roots, a single characteristic root does not provide enough independent solutions. Consider the recurrence

$$a_n = 4a_{n-1} - 4a_{n-2} \quad (n \geq 2).$$

Its characteristic polynomial is

$$p(t) = t^2 - 4t + 4 = (t - 2)^2,$$

which has only one root, $r = 2$, with multiplicity 2. If we used only pure powers of roots, we would write

$$a_n = A \cdot 2^n + B \cdot 2^n = (A + B)2^n = C2^n.$$

The two constants collapse into a single constant C . But an order-2 recurrence requires two independent initial values a_0 and a_1 to specify a sequence. For instance, if $a_0 = 1$ and $a_1 = 4$, the single constant gives $C = 1$ from a_0 , which forces $a_1 = 2 \neq 4$. One degree of freedom is missing.

We multiply the geometric sequence by n to generate a second linearly independent solution. Substituting $a_n = n2^n$ into the left-hand side of $a_n - 4a_{n-1} + 4a_{n-2} = 0$ gives

$$\begin{aligned} n2^n - 4(n-1)2^{n-1} + 4(n-2)2^{n-2} &= 2^{n-2}(4n - 8(n-1) + 4(n-2)) \\ &= 2^{n-2}(4n - 8n + 8 + 4n - 8) \\ &= 2^{n-2}(0) = 0. \end{aligned}$$

The sequence $n2^n$ is indeed a second linearly independent solution. The general solution for this recurrence is therefore $a_n = (A + Bn)2^n$.

The shift operator explains why polynomial factors appear systematically whenever a root is repeated.

Definition 5 (Shift operator). For a sequence $a = (a_n)_{n \geq 0}$, the **shift operator** E is defined by

$$(Ea)_n = a_{n+1}.$$

Thus $(E - r)a$ is the sequence whose n th term is $a_{n+1} - ra_n$.

Let $Q(n)$ be a polynomial in n . Applying the operator $E - r$ to the sequence $a_n = Q(n)r^n$ gives

$$\begin{aligned} ((E - r)a)_n &= Q(n+1)r^{n+1} - rQ(n)r^n \\ &= r^{n+1}(Q(n+1) - Q(n)). \end{aligned}$$

The difference

$$\Delta Q(n) = Q(n+1) - Q(n)$$

is the **forward difference** of Q . If $Q(n)$ has degree $q \geq 1$, then $\Delta Q(n)$ has degree $q - 1$ because the leading terms cancel:

$$(n+1)^q - n^q = qn^{q-1} + \dots$$

If Q is constant, then $\Delta Q = 0$.

Consequently, each application of $E - r$ reduces the degree of the polynomial factor multiplying r^n by one. Applying $E - r$ repeatedly gives

$$(E - r)^m(Q(n)r^n) = 0 \quad \text{whenever } \deg Q < m.$$

If r is a root of $p(t)$ of multiplicity m , then $p(t) = (t - r)^m q(t)$ for some polynomial $q(t)$. Because polynomials in the shift operator E commute, the recurrence operator factors as

$$p(E) = q(E)(E - r)^m.$$

Therefore, $p(E)$ annihilates every sequence of the form $Q(n)r^n$ with $\deg Q < m$. Choosing $Q(n) = 1, n, n^2, \dots, n^{m-1}$ yields the m linearly independent solutions

$$r^n, \quad nr^n, \quad n^2r^n, \quad \dots, \quad n^{m-1}r^n.$$

Theorem 6 (General homogeneous solution for real roots). Let the distinct real roots of the characteristic polynomial $p(t)$ be $r_1, r_2, \dots, r_k$ with respective multiplicities $m_1, m_2, \dots, m_k$, where $m_1 + m_2 + \dots + m_k = d$. Every solution of (1) has the form

$$a_n = \sum_{j=1}^k \left(A_{j,0} + A_{j,1}n + \dots + A_{j,m_j-1}n^{m_j-1} \right) r_j^n,$$

where the d numbers $A_{j,q}$ are constants determined by the initial conditions. In particular, a root r of multiplicity m contributes

$$(A_0 + A_1n + \dots + A_{m-1}n^{m-1}) r^n.$$

Example 7 (A repeated real root). Solve

$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \quad (n \geq 3)$$

with initial conditions

$$a_0 = 3, \quad a_1 = 3, \quad a_2 = 1.$$

Solution: Moving all terms to the left gives $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$. The characteristic polynomial factors as

$$p(t) = t^3 - 5t^2 + 8t - 4 = (t-1)(t-2)^2.$$

The root 1 is simple and contributes $A \cdot 1^n = A$. The root 2 has multiplicity 2 and contributes $(B + Cn)2^n$. The general solution is

$$a_n = A + (B + Cn)2^n.$$

Substituting the initial conditions produces the linear system:

$$\begin{aligned} n=0: & \quad A + B = 3, \\ n=1: & \quad A + 2B + 2C = 3, \\ n=2: & \quad A + 4B + 8C = 1. \end{aligned}$$

Subtracting the first equation from the second gives $B + 2C = 0$, so $B = -2C$. Subtracting the second equation from the third gives $2B + 6C = -2$. Substituting $B = -2C$ into this second relation yields

$$2(-2C) + 6C = 2C = -2 \implies C = -1.$$

Then $B = -2(-1) = 2$, and $A = 3 - B = 1$. The unique solution is

$$a_n = 1 + (2 - n)2^n.$$

Checking at $n = 3$, the formula gives $a_3 = 1 + (2 - 3)2^3 = 1 - 8 = -7$. The recurrence confirms:

$$a_3 = 5(1) - 8(3) + 4(3) = 5 - 24 + 12 = -7.$$

Example 8 (A triple root and successive differences). The recurrence

$$a_{n+3} - 6a_{n+2} + 12a_{n+1} - 8a_n = 0$$

has characteristic polynomial

$$p(t) = t^3 - 6t^2 + 12t - 8 = (t - 2)^3.$$

Its general solution is

$$a_n = (A + Bn + Cn^2)2^n.$$

The operator identity $(E - 2)a_n = 2^{n+1}\Delta Q(n)$ shows why $n^2 2^n$ is annihilated:

$$\begin{aligned} Q(n) = n^2 &\implies \Delta Q(n) = (n+1)^2 - n^2 = 2n+1, \\ &\implies \Delta^2 Q(n) = (2(n+1)+1) - (2n+1) = 2, \\ &\implies \Delta^3 Q(n) = 2 - 2 = 0. \end{aligned}$$

Thus $(E - 2)^3$ annihilates $n^2 2^n$. It also annihilates $n 2^n$ after two applications and 2^n after one. This explains why a root of multiplicity 3 accommodates polynomials of degree up to 2, but not a higher power such as $n^3 2^n$.

11.2.3 Summary of the homogeneous solution procedure

For a homogeneous linear recurrence with constant coefficients and specified initial values, the calculation follows four steps:

1. **Form the characteristic polynomial.** Move all sequence terms to one side, replace each term a_{n-j} by t^{d-j} , and set $p(t) = 0$.
2. **Find all roots and their multiplicities.** Factor $p(t)$ completely and record the multiplicity of each root.
3. **Write the general solution with unknown coefficients.** Each distinct real root r of multiplicity m contributes a polynomial in n of degree $m - 1$ multiplied by r^n .
4. **Fit the initial data.** Substitute the d given initial values into the general solution and solve the resulting system of linear equations for the unknown constants.

Initial conditions must always be applied to the complete general solution in Step 4, never to an incomplete form before all roots and their multiplicities are accounted for.

11.3 Nonhomogeneous polynomial–exponential driving terms

We now consider recurrences with an external driving term:

$$L[a]_n = a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_d a_{n-d} = P(n)r^n \quad (n \geq d), \quad (3)$$

where $L[a]_n$ denotes the linear recurrence operator, $P(n)$ is a polynomial in n , and $r \neq 0$ is a constant. The associated homogeneous recurrence is

$$L[a]_n = 0.$$

Theorem 9 (Particular plus homogeneous). If $a^{(p)}$ is any particular solution of $L[a]_n = P(n)r^n$, then

every solution of (3) can be written in the form

$$a_n = a_n^{(p)} + a_n^{(h)},$$

where $a^{(h)}$ is a solution of the associated homogeneous recurrence $L[a]_n = 0$.

Proof. Because L is a linear operator, substituting $a_n = a_n^{(p)} + a_n^{(h)}$ gives

$$L[a^{(p)} + a^{(h)}]_n = L[a^{(p)}]_n + L[a^{(h)}]_n = P(n)r^n + 0 = P(n)r^n.$$

Thus every sequence of this form is a solution.

Conversely, suppose a is an arbitrary solution of $L[a]_n = P(n)r^n$. Define the difference sequence $a^{(h)} = a - a^{(p)}$. By linearity,

$$L[a^{(h)}]_n = L[a - a^{(p)}]_n = L[a]_n - L[a^{(p)}]_n = P(n)r^n - P(n)r^n = 0.$$

Thus $a^{(h)}$ is indeed a solution to the homogeneous equation, so $a = a^{(p)} + a^{(h)}$. □

Theorem 2 establishes the solution procedure for nonhomogeneous recurrences:

1. Find one particular solution $a_n^{(p)}$ satisfying the nonhomogeneous equation, ignoring the initial conditions.
2. Write down the general solution $a_n^{(h)}$ of the associated homogeneous recurrence with unknown constants.
3. Form the complete sum $a_n = a_n^{(p)} + a_n^{(h)}$, and only then substitute the initial conditions to determine the constants.

Initial conditions must never be fitted to $a_n^{(h)}$ alone before adding $a_n^{(p)}$. The initial values $a_0, a_1, \dots$ belong to the total sequence a_n , so the particular solution shifts the values that the homogeneous part must match.

11.3.1 Choosing the trial form: undetermined coefficients

To find a particular solution for a driving term $P(n)r^n$, where $P(n)$ is a polynomial of degree ℓ , we use the method of undetermined coefficients. Let s be the multiplicity of r as a root of the characteristic polynomial $p(t)$. If r is not a root, we set $s = 0$. We select the trial form

$$a_n^{(p)} = n^s Q(n)r^n, \quad Q(n) = A_\ell n^\ell + A_{\ell-1} n^{\ell-1} + \dots + A_1 n + A_0. \quad (4)$$

The polynomial $Q(n)$ has the same degree ℓ as $P(n)$, with coefficients $A_0, A_1, \dots, A_\ell$ to be determined by substitution.

Two cases arise based on whether r is a characteristic root:

- **Nonresonance** ($s = 0$): If r is not a characteristic root, the trial form is simply $Q(n)r^n$.
- **Resonance** ($s \geq 1$): If r is a characteristic root of multiplicity s , the trial form must be multiplied by n^s .

The factor n^s is required when r is a characteristic root because the homogeneous operator L annihilates $r^n, nr^n, \dots, n^{s-1}r^n$. If we tried an unmodified form $Q(n)r^n$, the terms belonging to lower powers of n would produce zero rather than the driving term. Multiplying by n^s lifts every term in the trial sequence out of the homogeneous solution space.

Warning (The trial polynomial must include all lower-degree terms). If the driving polynomial $P(n)$ has degree ℓ , the trial polynomial $Q(n)$ must include every power from n^ℓ down to the constant term A_0 , even if some intermediate powers are missing in $P(n)$. For example, if the driving term is $(3n^2 - 1)4^n$, we must try

$$(An^2 + Bn + C)4^n.$$

Omitting the linear term Bn prevents matching coefficients because shifting the index from n to $n + 1$ expands $(n + 1)^2 = n^2 + 2n + 1$, generating linear terms during substitution.

11.3.2 Case 1: the exponential base is not a characteristic root

When the base r is not a root of $p(t)$, no term of the trial form $Q(n)r^n$ solves the homogeneous equation. This is the nonresonant case.

Example 10 (A nonresonant polynomial–exponential driving term). Solve

$$a_{n+2} - 3a_{n+1} + 2a_n = (2n + 1)3^n \quad (n \geq 0)$$

with initial conditions

$$a_0 = 0, \quad a_1 = 0.$$

Solution: First, solve the homogeneous equation $a_{n+2} - 3a_{n+1} + 2a_n = 0$. Its characteristic polynomial is

$$p(t) = t^2 - 3t + 2 = (t - 1)(t - 2),$$

which has roots 1 and 2. The homogeneous solution is

$$a_n^{(h)} = C_1 \cdot 1^n + C_2 \cdot 2^n = C_1 + C_2 2^n.$$

Next, find a particular solution. The driving term is $(2n + 1)3^n$, so the base is $r = 3$ and the polynomial $2n + 1$ has degree $\ell = 1$. Because 3 is not a root of $p(t)$, we set $s = 0$ and try

$$a_n^{(p)} = (An + B)3^n.$$

Substituting $a_n^{(p)}$ into the left-hand side of the recurrence gives

$$(A(n + 2) + B)3^{n+2} - 3(A(n + 1) + B)3^{n+1} + 2(An + B)3^n.$$

We factor out 3^n using $3^{n+2} = 9 \cdot 3^n$ and $3 \cdot 3^{n+1} = 9 \cdot 3^n$:

$$\begin{aligned} 3^n \left(9(A(n + 2) + B) - 9(A(n + 1) + B) + 2(An + B) \right) &= 3^n (9A(n + 2) - 9A(n + 1) + 2An + 2B) \\ &= 3^n (9A + 2An + 2B) \\ &= 3^n (2An + (9A + 2B)). \end{aligned}$$

Setting this equal to $(2n + 1)3^n$ and matching coefficients of n and the constant term yields

$$2A = 2, \quad 9A + 2B = 1.$$

Thus $A = 1$, and $9(1) + 2B = 1$ gives $B = -4$. A particular solution is

$$a_n^{(p)} = (n - 4)3^n.$$

The full solution is the sum of the particular and homogeneous solutions:

$$a_n = C_1 + C_2 2^n + (n - 4)3^n.$$

We now use the initial conditions $a_0 = 0$ and $a_1 = 0$:

$$\begin{aligned} n = 0 : \quad & C_1 + C_2 + (0 - 4)3^0 = C_1 + C_2 - 4 = 0, \\ n = 1 : \quad & C_1 + 2C_2 + (1 - 4)3^1 = C_1 + 2C_2 - 9 = 0. \end{aligned}$$

Thus $C_1 + C_2 = 4$ and $C_1 + 2C_2 = 9$. Subtracting the first equation from the second gives $C_2 = 5$, and so $C_1 = -1$. The required solution is

$$a_n = -1 + 5 \cdot 2^n + (n - 4)3^n.$$

Checking at $n = 2$, the recurrence gives

$$a_2 = 3a_1 - 2a_0 + (2(0) + 1)3^0 = 3(0) - 2(0) + 1 = 1.$$

The formula produces

$$a_2 = -1 + 5 \cdot 2^2 + (2 - 4)3^2 = -1 + 20 - 18 = 1.$$

11.3.3 Case 2: the exponential base is a characteristic root

When the base r is a characteristic root, the unmodified form $Q(n)r^n$ overlaps with the homogeneous solution family. This is called resonance. Multiplying by n^s , where s is the multiplicity of r , restores independence.

Example 11 (Resonance at a simple root). Solve

$$a_{n+2} - 3a_{n+1} + 2a_n = (n + 1)2^n \quad (n \geq 0)$$

with initial conditions

$$a_0 = 0, \quad a_1 = 1.$$

Solution: The characteristic polynomial is again $p(t) = (t - 1)(t - 2)$, with roots 1 and 2. The homogeneous solution is $a_n^{(h)} = C_1 + C_2 2^n$.

The driving term is $(n + 1)2^n$. The base $r = 2$ is a characteristic root of multiplicity $s = 1$. The unmodified trial $(An + B)2^n$ would fail: the term $B2^n$ is already a homogeneous solution and would vanish under substitution. We therefore multiply by $n^1 = n$:

$$a_n^{(p)} = n(An + B)2^n = (An^2 + Bn)2^n.$$

Let $q(n) = An^2 + Bn$. Substituting into the recurrence operator gives

$$q(n + 2)2^{n+2} - 3q(n + 1)2^{n+1} + 2q(n)2^n = 2^n(4q(n + 2) - 6q(n + 1) + 2q(n)).$$

Now compute each term inside the bracket:

$$\begin{aligned} 4q(n + 2) &= 4(A(n + 2)^2 + B(n + 2)) = 4An^2 + (16A + 4B)n + (16A + 8B), \\ -6q(n + 1) &= -6(A(n + 1)^2 + B(n + 1)) = -6An^2 - (12A + 6B)n - (6A + 6B), \\ 2q(n) &= 2(An^2 + Bn) = 2An^2 + 2Bn. \end{aligned}$$

Summing these expressions, the coefficient of n^2 vanishes:

$$(4 - 6 + 2)An^2 = 0.$$

This cancellation is expected: the n^2 term cancels precisely because 2 is a characteristic root. The remaining terms are

$$(16A + 4B - 12A - 6B + 2B)n + (16A + 8B - 6A - 6B) = 4An + (10A + 2B).$$

Equating $2^n(4An + 10A + 2B)$ with $(n + 1)2^n$ yields

$$4A = 1, \quad 10A + 2B = 1.$$

Thus $A = 1/4$, and $10(1/4) + 2B = 1$ gives $2B = 1 - 5/2 = -3/2$, so $B = -3/4$. A particular solution is

$$a_n^{(p)} = \left(\frac{1}{4}n^2 - \frac{3}{4}n \right) 2^n = \frac{n(n-3)}{4} 2^n.$$

The full solution is

$$a_n = C_1 + C_2 2^n + \frac{n(n-3)}{4} 2^n.$$

We fit the initial conditions $a_0 = 0$ and $a_1 = 1$:

$$\begin{aligned} n = 0 : \quad C_1 + C_2 &= 0, \\ n = 1 : \quad C_1 + 2C_2 + \frac{1(1-3)}{4} \cdot 2 &= C_1 + 2C_2 - 1 = 1. \end{aligned}$$

Thus $C_1 + C_2 = 0$ and $C_1 + 2C_2 = 2$. Subtracting yields $C_2 = 2$, which gives $C_1 = -2$. The solution is

$$a_n = -2 + 2 \cdot 2^n + \frac{n(n-3)}{4} 2^n.$$

Checking at $n = 2$, the recurrence gives

$$a_2 = 3a_1 - 2a_0 + (0 + 1)2^0 = 3(1) - 0 + 1 = 4.$$

The formula produces

$$a_2 = -2 + 2 \cdot 4 + \frac{2(-1)}{4} \cdot 4 = -2 + 8 - 2 = 4.$$

Example 12 (Resonance at a double root). Solve

$$a_{n+2} - 4a_{n+1} + 4a_n = 2^n \quad (n \geq 0)$$

with initial conditions $a_0 = 0$ and $a_1 = 0$.

Solution: The characteristic polynomial is $p(t) = t^2 - 4t + 4 = (t - 2)^2$. The root 2 has multiplicity 2, so the homogeneous solution is

$$a_n^{(h)} = (C_1 + C_2 n) 2^n.$$

The driving term is $2^n = 1 \cdot 2^n$, where the polynomial has degree $\ell = 0$ (a constant) and the base $r = 2$ has multiplicity $s = 2$. Both 2^n and $n2^n$ are homogeneous solutions, so we must multiply by n^2 :

$$a_n^{(p)} = An^2 2^n.$$

Substituting into the recurrence operator gives

$$\begin{aligned} A(n+2)^2 2^{n+2} - 4A(n+1)^2 2^{n+1} + 4An^2 2^n &= A2^n (4(n+2)^2 - 8(n+1)^2 + 4n^2) \\ &= A2^n (4(n^2 + 4n + 4) - 8(n^2 + 2n + 1) + 4n^2) \\ &= A2^n ((4 - 8 + 4)n^2 + (16 - 16)n + (16 - 8)) \\ &= 8A2^n. \end{aligned}$$

Both the n^2 and n terms cancel, leaving $8A2^n = 2^n$. Hence $8A = 1$, so $A = 1/8$. The particular solution is

$$a_n^{(p)} = \frac{1}{8}n^22^n.$$

The general solution is

$$a_n = (C_1 + C_2n)2^n + \frac{1}{8}n^22^n.$$

Fitting the initial conditions $a_0 = 0$ and $a_1 = 0$:

$$n = 0 : \quad C_1 = 0,$$

$$n = 1 : \quad (C_1 + C_2)2 + \frac{1}{8}(1)2 = 2C_2 + \frac{1}{4} = 0 \implies C_2 = -\frac{1}{8}.$$

The solution is

$$a_n = \left(-\frac{1}{8}n + \frac{1}{8}n^2\right)2^n = \frac{n(n-1)}{8}2^n.$$

Checking at $n = 2$, the recurrence gives $a_2 = 4a_1 - 4a_0 + 2^0 = 1$, and the formula gives $\frac{2(1)}{8} \cdot 4 = 1$. At $n = 3$, the recurrence gives $a_3 = 4(1) - 4(0) + 2^1 = 6$, and the formula gives $\frac{3(2)}{8} \cdot 8 = 6$.

11.4 Complex characteristic roots

A polynomial with real coefficients need not factor completely into real linear factors. For example, the equation

$$t^2 - 2t + 2 = 0$$

has roots $1 + i$ and $1 - i$, where i satisfies $i^2 = -1$. When a linear recurrence with real coefficients has nonreal characteristic roots, every sequence value generated from real initial conditions is a real number. We use the polar form of complex numbers to convert powers of complex roots into sines and cosines, yielding a real-valued formula.

11.4.1 Complex numbers, radius, and argument

A **complex number** is written in Cartesian form as

$$z = x + iy,$$

where $x, y \in \mathbb{R}$. The real number x is the real part $\text{Re}(z)$, y is the imaginary part $\text{Im}(z)$, and

$$\bar{z} = x - iy$$

is the **complex conjugate** of z .

For $z \neq 0$, the **radius** (or modulus) of z is its distance from the origin in the complex plane:

$$\rho = |z| = \sqrt{x^2 + y^2}.$$

An **argument** of z is an angle θ measured counterclockwise from the positive real axis satisfying

$$\cos \theta = \frac{x}{\rho}, \quad \sin \theta = \frac{y}{\rho}.$$

Euler's formula states that

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Therefore, z can be written in **polar form** as

$$z = \rho(\cos \theta + i \sin \theta) = \rho e^{i\theta}.$$

Taking powers in polar form is governed by De Moivre's formula:

$$z^n = \rho^n (\cos(n\theta) + i \sin(n\theta)). \quad (5)$$

Raising z to the n th power scales its distance from the origin by ρ^n and multiplies its angle by n .

Example 13 (Computing a complex power in polar form). Compute z^5 for $z = -1 + i\sqrt{3}$.

Solution: The radius of z is

$$\rho = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2.$$

The point $(-1, \sqrt{3})$ lies in the second quadrant of the complex plane, with

$$\cos \theta = -\frac{1}{2}, \quad \sin \theta = \frac{\sqrt{3}}{2}.$$

We choose the argument $\theta = 2\pi/3$. The polar form of z is

$$z = 2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right).$$

By De Moivre's formula (5),

$$\begin{aligned} z^5 &= 2^5 \left(\cos \frac{10\pi}{3} + i \sin \frac{10\pi}{3} \right) \\ &= 32 \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) \\ &= 32 \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) \\ &= -16 - 16i\sqrt{3}. \end{aligned}$$

Polar form allows us to compute z^5 without expanding the product $(-1 + i\sqrt{3})^5$ term by term.

11.4.2 Conjugate roots produce real sine and cosine terms

Because the coefficients of the characteristic polynomial $p(t)$ are real, any nonreal root $z = x + iy$ must be accompanied by its complex conjugate $\bar{z} = x - iy$ with the same multiplicity. In polar form,

$$z = \rho(\cos \theta + i \sin \theta), \quad \bar{z} = \rho(\cos \theta - i \sin \theta).$$

De Moivre's formula gives expressions for the powers of both roots:

$$\begin{aligned} z^n &= \rho^n (\cos(n\theta) + i \sin(n\theta)), \\ \bar{z}^n &= \rho^n (\cos(n\theta) - i \sin(n\theta)). \end{aligned}$$

Both $(z^n)_{n \geq 0}$ and $(\bar{z}^n)_{n \geq 0}$ are solutions to the recurrence. Because the recurrence is linear and homogeneous, any linear combination of solutions is again a solution. Adding and subtracting these two equations produces two purely real solutions:

$$\frac{z^n + \bar{z}^n}{2} = \rho^n \cos(n\theta), \quad \frac{z^n - \bar{z}^n}{2i} = \rho^n \sin(n\theta).$$

Both sequences are real-valued. Forming an arbitrary real linear combination of them gives

$$\rho^n (A \cos(n\theta) + B \sin(n\theta)),$$

where A and B are real constants.

This expression reveals the geometric role of the polar coordinates:

- The radius factor ρ^n controls the amplitude growth or decay: if $\rho > 1$, the terms grow exponentially; if $\rho < 1$, they decay to zero; if $\rho = 1$, the amplitude remains constant.
- The trigonometric factors $\cos(n\theta)$ and $\sin(n\theta)$ cause the terms to oscillate in sign and value, with a period determined by the angle θ .

Theorem 14 (Full real form of the homogeneous solution). Let the characteristic polynomial of an order- d linear recurrence have real roots $r_1, \dots, r_k$ with multiplicities $m_1, \dots, m_k$, and nonreal conjugate pairs $z_j, \bar{z}_j$ with multiplicities s_j , where

$$z_j = \rho_j (\cos \theta_j + i \sin \theta_j) \quad (j = 1, \dots, q).$$

Then every real-valued solution of the recurrence has the form

$$a_n = \sum_{j=1}^k \left(\sum_{h=0}^{m_j-1} A_{j,h} n^h \right) r_j^n + \sum_{j=1}^q \rho_j^n \left[\left(\sum_{h=0}^{s_j-1} B_{j,h} n^h \right) \cos(n\theta_j) + \left(\sum_{h=0}^{s_j-1} C_{j,h} n^h \right) \sin(n\theta_j) \right],$$

where all coefficients $A_{j,h}, B_{j,h}, C_{j,h}$ are real constants.

Each real root of multiplicity m contributes m arbitrary constants. Each conjugate pair of multiplicity s contributes $2s$ arbitrary constants: s constants for the cosine terms and s constants for the sine terms. The total number of constants equals the degree d of the characteristic polynomial, matching the d initial conditions needed to specify a unique sequence.

Example 15 (Solving a recurrence with complex roots). Solve

$$a_{n+2} = 2a_{n+1} - 2a_n \quad (n \geq 0)$$

with initial conditions

$$a_0 = 2, \quad a_1 = 0.$$

Solution: Moving all terms to the left gives $a_{n+2} - 2a_{n+1} + 2a_n = 0$. The characteristic equation is

$$t^2 - 2t + 2 = 0.$$

By the quadratic formula,

$$t = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

We convert $z = 1 + i$ to polar form:

$$\rho = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Because $\cos \theta = 1/\sqrt{2}$ and $\sin \theta = 1/\sqrt{2}$, we choose $\theta = \pi/4$.

The two roots form a single simple conjugate pair, so the real general solution is

$$a_n = (\sqrt{2})^n \left(A \cos \frac{n\pi}{4} + B \sin \frac{n\pi}{4} \right).$$

We use the initial conditions $a_0 = 2$ and $a_1 = 0$ to find A and B :

$$\begin{aligned} n = 0 : \quad a_0 &= (\sqrt{2})^0 (A \cos 0 + B \sin 0) = A = 2, \\ n = 1 : \quad a_1 &= (\sqrt{2})^1 \left(A \cos \frac{\pi}{4} + B \sin \frac{\pi}{4} \right) = \sqrt{2} \left(2 \cdot \frac{1}{\sqrt{2}} + B \cdot \frac{1}{\sqrt{2}} \right) = 2 + B = 0. \end{aligned}$$

Thus $A = 2$ and $B = -2$. The solution is

$$a_n = 2(\sqrt{2})^n \left(\cos \frac{n\pi}{4} - \sin \frac{n\pi}{4} \right).$$

We check the first few terms against the recurrence:

- From the recurrence:

$$\begin{aligned} a_2 &= 2a_1 - 2a_0 = 2(0) - 2(2) = -4, \\ a_3 &= 2a_2 - 2a_1 = 2(-4) - 2(0) = -8, \\ a_4 &= 2a_3 - 2a_2 = 2(-8) - 2(-4) = -8, \\ a_5 &= 2a_4 - 2a_3 = 2(-8) - 2(-8) = 0, \\ a_6 &= 2a_5 - 2a_4 = 2(0) - 2(-8) = 16. \end{aligned}$$

- From the formula:

$$\begin{aligned} a_2 &= 2(\sqrt{2})^2 \left(\cos \frac{\pi}{2} - \sin \frac{\pi}{2} \right) = 2(2)(0 - 1) = -4, \\ a_3 &= 2(\sqrt{2})^3 \left(\cos \frac{3\pi}{4} - \sin \frac{3\pi}{4} \right) = 2(2\sqrt{2}) \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) = 4\sqrt{2}(-\sqrt{2}) = -8, \\ a_4 &= 2(\sqrt{2})^4 (\cos \pi - \sin \pi) = 2(4)(-1 - 0) = -8, \\ a_5 &= 2(\sqrt{2})^5 \left(\cos \frac{5\pi}{4} - \sin \frac{5\pi}{4} \right) = 2(4\sqrt{2}) \left(-\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}} \right) \right) = 0, \\ a_6 &= 2(\sqrt{2})^6 \left(\cos \frac{3\pi}{2} - \sin \frac{3\pi}{2} \right) = 2(8)(0 - (-1)) = 16. \end{aligned}$$

The formula matches the recurrence values exactly. The sequence values $(2, 0, -4, -8, -8, 0, 16, \dots)$ exhibit oscillatory changes in sign and value driven by the trigonometric factors, enclosed within an exponential envelope growing by a factor of $\sqrt{2}$ per step.

11.5 A complete solution checklist

To solve any linear constant-coefficient recurrence with specified initial conditions, follow these steps in order:

1. **Classify the recurrence.** Confirm that the recurrence is linear with constant coefficients, and determine its order d .
2. **Form the characteristic polynomial.** Move all homogeneous sequence terms to one side, replace each term a_{n-j} by t^{d-j} , and write $p(t) = 0$.
3. **Find all roots and multiplicities.** Factor $p(t)$ completely. Identify real roots and their multiplicities, and express any nonreal roots as conjugate pairs in polar form $z = \rho(\cos \theta \pm i \sin \theta)$.
4. **Write the general homogeneous solution.**

- Each real root r of multiplicity m contributes $(A_0 + A_1n + \cdots + A_{m-1}n^{m-1})r^n$.
 - Each complex conjugate pair of multiplicity s contributes $\rho^n(P(n)\cos(n\theta) + Q(n)\sin(n\theta))$, where P and Q are polynomials of degree $s - 1$.
5. **Find a particular solution if nonhomogeneous.** For a driving term $P(n)r^n$ with $\deg P = \ell$, use the trial form $a_n^{(p)} = n^s Q(n)r^n$, where $\deg Q = \ell$ and s is the multiplicity of r as a characteristic root ($s = 0$ if r is not a root). Substitute into the recurrence and solve for the unknown coefficients of Q .
 6. **Add the solutions.** Form the full solution $a_n = a_n^{(p)} + a_n^{(h)}$.
 7. **Fit the initial data.** Substitute the d initial values into a_n and solve the resulting linear system for the unknown constants in $a_n^{(h)}$.
 8. **Verify the solution.** Test the resulting formula against the initial values and at least one subsequent term computed directly from the recurrence.

11.6 External References

11.6.1 Additional material (optional)

- [Discrete Math II - 8.2.1 Solving Linear Homogeneous Recurrence Relations](#)
- [Discrete Math II - 8.2.2 Repeated Roots in Recurrence Relations](#)
- [Discrete Math II - 8.2.4 Solving Nonhomogeneous Recurrence Relations](#)
- [TrevTutor - Solving Linear Recurrence Relations](#)
- [Discrete Math II - 8.1.1 Applications of Recurrence Relations](#)