

5. Inclusion–exclusion and applications

Combinatorics

Fall 2026

Suppose we want to count the students taking algebra or geometry. Adding the two course enrollments counts every student in both courses twice. With three or more courses, a student may appear in several intersections, and each kind of overlap requires its own correction. The **principle of inclusion–exclusion** organizes these corrections.

We first develop inclusion–exclusion for unions and then rewrite it for objects that must avoid several violations. The applications follow the order in which they arose in lecture: bounded integer solutions, forbidden substrings, Euler’s totient function, a grouped formula for symmetric conditions, and derangements. Graphs without isolated vertices, full-alphabet strings, and surjective functions provide further applications of the same formula.

After studying these notes, you will be able to do the following.

- Explain and apply the two-set and three-set inclusion–exclusion formulas.
- State the general principle of inclusion–exclusion for m finite sets and compute sizes of unions.
- Use indicator functions to prove the inclusion–exclusion formula.
- Express problems with multiple constraints using the complement (violation-set) form.
- Count integer solutions to linear equations with upper bounds by combining inclusion–exclusion with stars and bars.
- Count strings without repeated letters that avoid prescribed substrings, including cases in which substring conditions are incompatible.
- Derive the product formula for Euler’s totient function.
- Recognize when the count for imposed conditions depends only on the number of conditions and apply the grouped inclusion–exclusion formula.
- Count labeled graphs with no isolated vertices.
- Derive the formula for derangements D_n and count permutations with a specified number of fixed points.
- Count surjective functions and strings that use every symbol of an alphabet.

5.1 The geometry of overlapping sets

Let A and B be finite subsets of a universal set U . The union $A \cup B$ consists of all elements belonging to A , to B , or to both. When $A \cap B = \emptyset$, the sum rule gives $|A \cup B| = |A| + |B|$. When $A \cap B \neq \emptyset$, every element $x \in A \cap B$ contributes 1 to $|A|$ and 1 to $|B|$, so the sum $|A| + |B|$ counts each element of $A \cap B$. Subtracting $|A \cap B|$ removes this extra copy.

Theorem 1 (Inclusion–exclusion for two sets). For any finite sets A and B ,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Example 2 (Students enrolled in two courses). In a department of mathematics, 30 students take abstract algebra, 25 take geometry, and 8 take both courses. How many students take at least one of these two courses?

Solution: Let A be the set of students taking algebra and G the set of students taking geometry. The set of students taking at least one course is $A \cup G$. By [Theorem 1](#),

$$|A \cup G| = |A| + |G| - |A \cap G| = 30 + 25 - 8 = 47.$$

The 8 students enrolled in both courses were counted in $|A|$ and in $|G|$; subtracting 8 leaves each student counted exactly once.

Now consider three finite sets A , B , and C . If we begin with the sum of their individual sizes, $|A| + |B| + |C|$, an element belonging to exactly one set is counted once. An element belonging to the intersection of two sets (say $A \cap B$, but not C) is counted twice: once in $|A|$ and once in $|B|$. An element belonging to all three sets, $A \cap B \cap C$, is counted three times.

To correct for the pairwise overlaps, we subtract the sizes of all three two-way intersections:

$$-|A \cap B| - |A \cap C| - |B \cap C|.$$

An element in exactly two sets was counted twice and is now subtracted once, leaving a net count of $2 - 1 = 1$. However, an element in all three sets was counted three times in the initial sum and subtracted three times (once in each pairwise intersection). Its net count is now $3 - 3 = 0$. To count such elements in the union, we must add back the size of the triple intersection $|A \cap B \cap C|$.

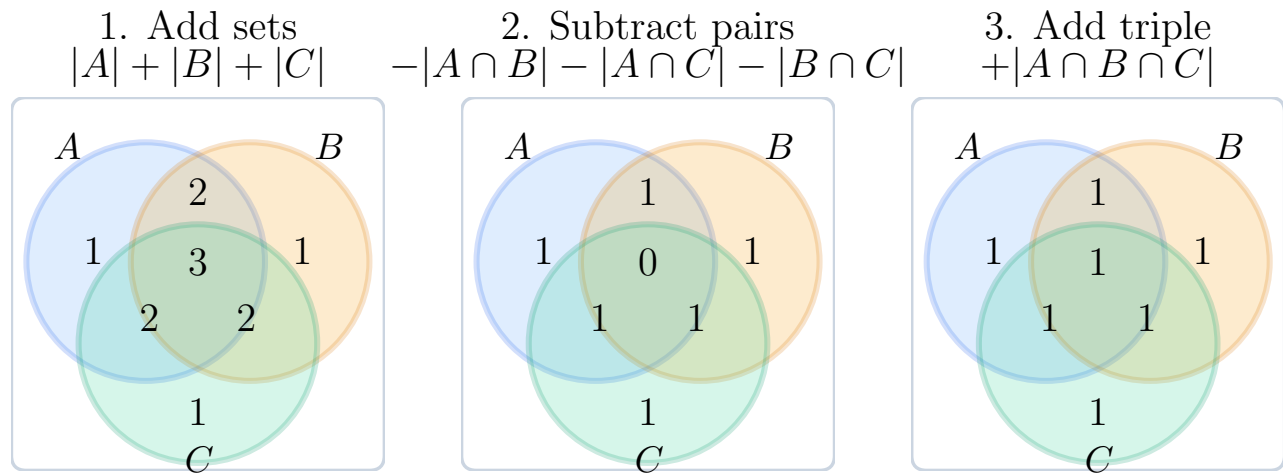


Figure 1: Multiplicities in each Venn region across the three stages of inclusion–exclusion for three sets. Stage 1 adds individual sizes; Stage 2 subtracts pairwise intersections; Stage 3 adds the triple intersection. *Figure description:* Three side-by-side Venn diagrams for sets A , B , and C . In the first, regions belonging to exactly one, two, or three sets are labeled with multiplicities 1, 2, and 3. In the second, after subtracting pairwise intersections, they are labeled 1, 1, and 0. In the third, after adding the triple intersection, all seven regions inside the union are labeled 1.

[Figure 1](#) tracks how each step adjusts the multiplicity of every region in the Venn diagram. The net result is that every element belonging to at least one of the three sets has a final coefficient of 1.

Theorem 3 (Inclusion–exclusion for three sets). For any finite sets A , B , and C ,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Example 4 (Course enrollments in three subjects). A survey of 114 mathematics majors records course enrollments as follows: 30 students take algebra, 25 take geometry, and 50 take calculus. Furthermore, 8 take algebra and geometry, 15 take algebra and calculus, 12 take geometry and calculus, and 5 take all three courses.

1. How many students take at least one of these three courses?
2. How many students take none of these three courses?

Solution: Let A , G , and C denote the sets of students taking algebra, geometry, and calculus, respectively.

1. The set of students taking at least one course is $A \cup G \cup C$. Applying [Theorem 3](#):

$$\begin{aligned} |A \cup G \cup C| &= (|A| + |G| + |C|) - (|A \cap G| + |A \cap C| + |G \cap C|) + |A \cap G \cap C| \\ &= (30 + 25 + 50) - (8 + 15 + 12) + 5 \\ &= 105 - 35 + 5 = 75. \end{aligned}$$

2. The universal set U has $|U| = 114$ students. The students taking none of the three courses form the complement $U \setminus (A \cup G \cup C)$. By the difference rule,

$$|U \setminus (A \cup G \cup C)| = |U| - |A \cup G \cup C| = 114 - 75 = 39.$$

5.2 The general principle of inclusion–exclusion

To state the principle for any finite number of sets $A_1, A_2, \dots, A_m$, we index the intersections by subsets of the index set $\{1, 2, \dots, m\}$. The size $|I|$ determines whether the term is added or subtracted: terms with odd $|I|$ carry a positive sign, while terms with even $|I|$ carry a negative sign.

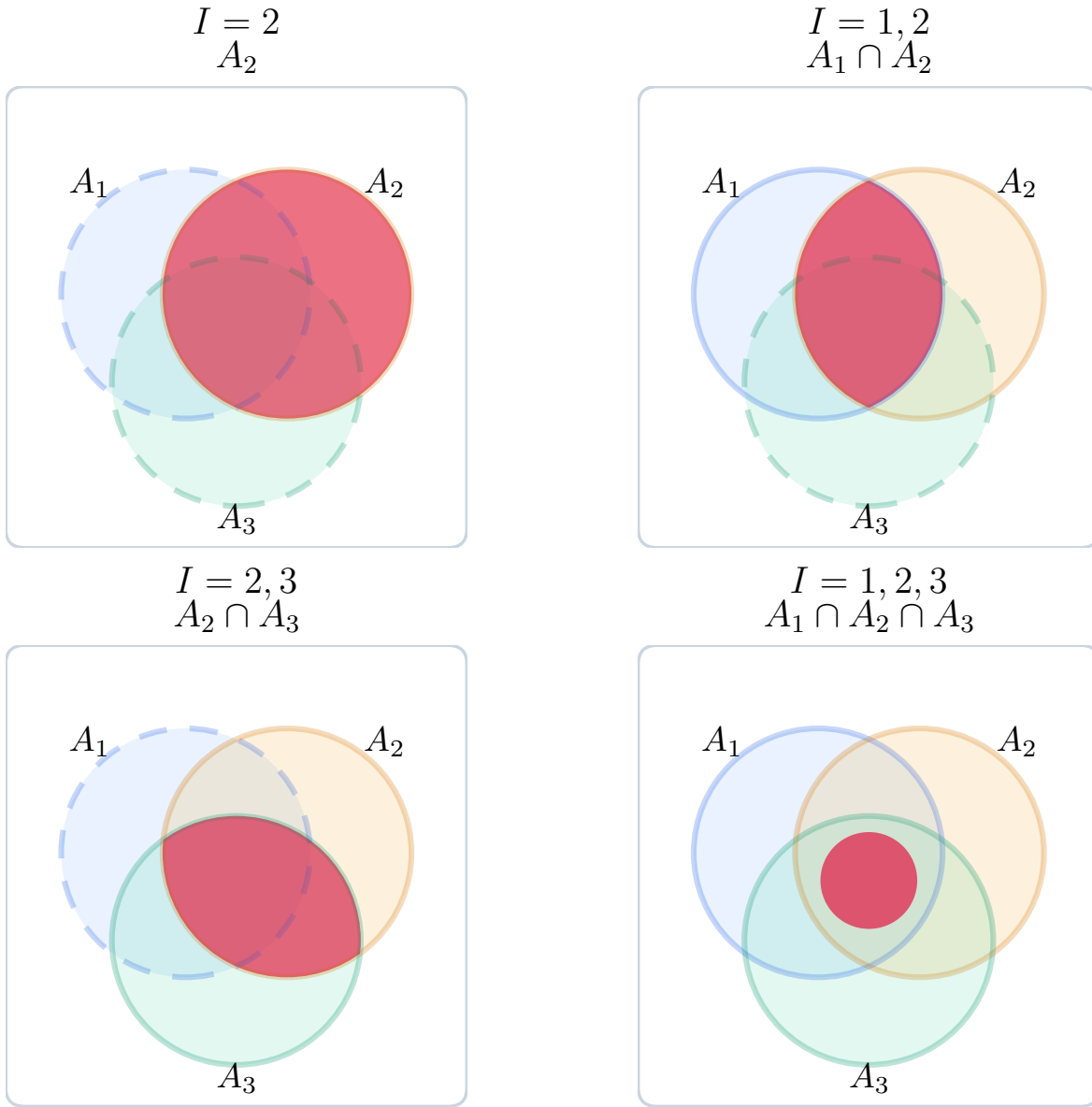


Figure 2: Four intersections corresponding to index subsets of $\{1, 2, 3\}$: $I = \{2\}$ gives A_2 ; $I = \{1, 2\}$ gives $A_1 \cap A_2$; $I = \{2, 3\}$ gives $A_2 \cap A_3$; and $I = \{1, 2, 3\}$ gives $A_1 \cap A_2 \cap A_3$.

Figure description: A two-by-two grid of Venn diagrams for three sets A_1 , A_2 , and A_3 , showing highlighted regions for A_2 , A_1 intersect A_2 , A_2 intersect A_3 , and the triple intersection A_1 intersect A_2 intersect A_3 .

Theorem 5 (Principle of inclusion–exclusion). Let $A_1, A_2, \dots, A_m$ be finite sets. Then

$$\left| \bigcup_{i=1}^m A_i \right| = \sum_{\emptyset \neq I \subseteq \{1, 2, \dots, m\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|.$$

Grouping the index subsets by their size $k = |I|$ gives an equivalent expanded form:

$$\left| \bigcup_{i=1}^m A_i \right| = \sum_{k=1}^m (-1)^{k+1} \sum_{\substack{I \subseteq \{1, \dots, m\} \\ |I|=k}} \left| \bigcap_{i \in I} A_i \right|.$$

The inner sum for $k = 1$ adds the sizes of all individual sets; for $k = 2$, it subtracts all $C(m, 2)$ pairwise

intersections; for $k = 3$, it adds all $C(m, 3)$ triple intersections, continuing up to $k = m$.

5.2.1 Proof via indicator functions

A clean and general proof of [Theorem 5](#) uses indicator functions, also called **characteristic functions**. Let U be a finite universal set containing $A_1, \dots, A_m$. For any subset $A \subseteq U$, the **indicator function** $\mathbf{1}_A : U \rightarrow \{0, 1\}$ is defined by

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Indicator functions satisfy three basic algebraic properties:

1. The cardinality of A is the sum of its indicator values over the universe:

$$|A| = \sum_{x \in U} \mathbf{1}_A(x).$$

2. The indicator of an intersection is the product of the indicators, since an element belongs to an intersection precisely when it belongs to every set in the intersection:

$$\mathbf{1}_{\bigcap_{i \in I} A_i}(x) = \prod_{i \in I} \mathbf{1}_{A_i}(x).$$

3. The indicator of the complement $U \setminus A$ is

$$\mathbf{1}_{U \setminus A}(x) = 1 - \mathbf{1}_A(x).$$

Proof of Theorem 5. An element $x \in U$ does not belong to the union $\bigcup_{i=1}^m A_i$ if and only if $x \notin A_i$ for every $i \in \{1, \dots, m\}$. Therefore,

$$\mathbf{1}_{U \setminus \bigcup_{i=1}^m A_i}(x) = \prod_{i=1}^m \mathbf{1}_{U \setminus A_i}(x) = \prod_{i=1}^m (1 - \mathbf{1}_{A_i}(x)).$$

Using the complement property $\mathbf{1}_{U \setminus A_i}(x) = 1 - \mathbf{1}_{A_i}(x)$, we have

$$\mathbf{1}_{U \setminus \bigcup_{i=1}^m A_i}(x) = 1 - \prod_{i=1}^m \mathbf{1}_{A_i}(x).$$

We expand the product $\prod_{i=1}^m (1 - y_i)$ with $y_i = \mathbf{1}_{A_i}(x)$. Every term in the expansion corresponds to choosing $-y_i$ for a subset of indices $I \subseteq \{1, \dots, m\}$ and 1 for the remaining indices:

$$\prod_{i=1}^m (1 - \mathbf{1}_{A_i}(x)) = \sum_{I \subseteq \{1, \dots, m\}} (-1)^{|I|} \prod_{i \in I} \mathbf{1}_{A_i}(x) = 1 + \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|} \mathbf{1}_{\bigcap_{i \in I} A_i}(x).$$

Subtracting this product from 1 yields

$$\mathbf{1}_{\bigcup_{i=1}^m A_i}(x) = \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} \mathbf{1}_{\bigcap_{i \in I} A_i}(x).$$

Finally, summing this identity over all $x \in U$ and using the identity $|B| = \sum_{x \in U} \mathbf{1}_B(x)$ for each term gives

$$\left| \bigcup_{i=1}^m A_i \right| = \sum_{\emptyset \neq I \subseteq \{1, \dots, m\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right|,$$

which proves the theorem. □

5.2.2 The complement and violation-set form

In combinatorial applications, we frequently want to count objects in a universe U that satisfy several conditions simultaneously. Rather than working directly with the satisfying objects, it is often simpler to identify the **violation sets** $A_1, A_2, \dots, A_m$, where $A_i \subseteq U$ consists of the objects that fail condition i . An object satisfies all m conditions if and only if it belongs to none of the violation sets. The set of valid objects is therefore the complement of the union of the violation sets:

$$U \setminus \bigcup_{i=1}^m A_i.$$

Subtracting the union formula from $|U|$ gives the complement form of inclusion–exclusion:

$$\left| U \setminus \bigcup_{i=1}^m A_i \right| = |U| - \left| \bigcup_{i=1}^m A_i \right| = \sum_{I \subseteq \{1, \dots, m\}} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|, \quad (1)$$

where the term for $I = \emptyset$ is defined to be $|U|$.

5.3 Integer solutions with upper bounds

The method of stars and bars counts nonnegative integer solutions to equations of the form $x_1 + x_2 + \dots + x_r = N$. Lower bounds $x_i \geq \ell_i$ are easily handled by a change of variables $x'_i = x_i - \ell_i$. Upper bounds $x_i \leq q_i$, however, cannot be incorporated by a simple variable shift because solutions may violate the bound in one coordinate, two coordinates, or more. Inclusion–exclusion provides the mechanism to enforce upper bounds.

Example 6 (Upper bounds on three of four variables). How many integer solutions satisfy

$$x_1 + x_2 + x_3 + x_4 = 30, \quad x_i \geq 0 \quad (1 \leq i \leq 4),$$

with $x_1 \leq 9$, $x_2 \leq 7$, and $x_3 \leq 4$? There is no upper bound on x_4 .

Outline: Let U contain all nonnegative solutions of the equation, before imposing the three upper bounds. Let

$$A_1 = \{x_1 \geq 10\}, \quad A_2 = \{x_2 \geq 8\}, \quad A_3 = \{x_3 \geq 5\},$$

where each set is understood to contain solutions in U satisfying the displayed violation. The requested solutions lie in $U \setminus (A_1 \cup A_2 \cup A_3)$.

We will compute the cardinality of U , then the cardinality of each of the three single intersections, each of the three pairwise intersections, and the triple intersection. The complement form of inclusion–exclusion will combine those eight numbers.

Computing the cardinalities: Stars and bars gives

$$|U| = C(30 + 4 - 1, 4 - 1) = C(33, 3) = 5456.$$

In A_1 , set $y_1 = x_1 - 10$. The transformed equation is

$$y_1 + x_2 + x_3 + x_4 = 20,$$

so $|A_1| = C(23, 3) = 1771$. For A_2 , subtract 8 from x_2 ; for A_3 , subtract 5 from x_3 . The three single-intersection counts are

- $|A_1| = C(23, 3) = 1771$;
- $|A_2| = C(25, 3) = 2300$;
- $|A_3| = C(28, 3) = 3276$.

For $A_1 \cap A_2$, set $y_1 = x_1 - 10$ and $y_2 = x_2 - 8$. We obtain

$$y_1 + y_2 + x_3 + x_4 = 12,$$

and therefore $|A_1 \cap A_2| = C(15, 3) = 455$. Applying the corresponding two shifts in each of the other intersections gives

- $|A_1 \cap A_2| = C(15, 3) = 455$;
- $|A_1 \cap A_3| = C(18, 3) = 816$;
- $|A_2 \cap A_3| = C(20, 3) = 1140$.

Finally, in $A_1 \cap A_2 \cap A_3$ we subtract 10, 8, and 5 from x_1 , x_2 , and x_3 , respectively. The remaining total is $30 - 10 - 8 - 5 = 7$, so

$$|A_1 \cap A_2 \cap A_3| = C(10, 3) = 120.$$

Inclusion–exclusion now gives

$$\begin{aligned} |U \setminus (A_1 \cup A_2 \cup A_3)| &= 5456 - (1771 + 2300 + 3276) \\ &\quad + (455 + 816 + 1140) - 120 \\ &= 400. \end{aligned}$$

No binomial coefficient was used to abbreviate the choices of violation sets; every intersection appears in the calculation.

The answer has a direct check. We may choose each of x_1, x_2, x_3 independently within its own range. There are 10, 8, and 5 choices, respectively. Then $x_4 = 30 - x_1 - x_2 - x_3$ is uniquely determined and is at least $30 - 9 - 7 - 4 = 10$, so it is nonnegative. This gives $10 \cdot 8 \cdot 5 = 400$ solutions.

5.4 Strings with forbidden substrings

When counting arrangements of distinct items that must avoid specific consecutive patterns, we combine the block-reduction technique with inclusion–exclusion.

To enforce that a specific substring appears, we treat that entire sequence of symbols as a single compound object. When two requested substrings share a letter, that one letter must occupy the same position in both occurrences if repetition is forbidden. The conditions may merge into a longer block, or they may be incompatible.

Example 7 (Four substrings with compatible and incompatible pairs). How many length-10 strings can be formed from the 26 uppercase English letters if no letter may be repeated and none of the following strings occurs as a substring?

CAST, TUG, GAS, SAG

Solution: Let U be the set of all length-10 strings with no repeated letter. Thus $|U| = P(26, 10)$. Define four violation sets:

- A_1 : strings containing CAST;
- A_2 : strings containing TUG;
- A_3 : strings containing GAS;
- A_4 : strings containing SAG.

We first count the individual violation sets. A copy of **CAST** can start in any of the first 7 positions. After placing it, we fill the other 6 positions with distinct letters chosen in order from the remaining 22 letters. A three-letter substring has 8 possible starting positions, after which the other 7 positions are filled from the remaining 23 letters. Because letters cannot repeat, none of the specified substrings can occur twice in the same string. The start-position cases are therefore disjoint, and

$$\begin{aligned} |A_1| &= 7P(22, 6), \\ |A_2| &= 8P(23, 7), \\ |A_3| &= 8P(23, 7), \\ |A_4| &= 8P(23, 7). \end{aligned}$$

The pairwise intersections are not all alike. We examine all six.

- The conditions **CAST** and **TUG** share their letter **T**. The only compatible placement joins them into the block **CASTUG**. This length-6 block has 5 possible starting positions; the remaining 4 positions use distinct letters from the other 20. Hence

$$|A_1 \cap A_2| = 5P(20, 4).$$

- The conditions **CAST** and **GAS** cannot both hold. Both would use the same adjacent pair **AS**, but **CAST** requires **C** immediately before that pair while **GAS** requires **G** there. Thus $|A_1 \cap A_3| = 0$.
- The conditions **CAST** and **SAG** cannot both hold: **CAST** places **A** before **S**, while **SAG** places the same **S** before the same **A**. Thus $|A_1 \cap A_4| = 0$.
- The conditions **TUG** and **GAS** join at their common letter **G** to form **TUGAS**. This length-5 block has 6 possible starting positions, and the remaining 5 positions use distinct letters from the other 21. Hence

$$|A_2 \cap A_3| = 6P(21, 5).$$

- The conditions **TUG** and **SAG** both end in their only copy of **G**. They would require both **U** and **A** immediately before that **G**, so $|A_2 \cap A_4| = 0$.
- The conditions **GAS** and **SAG** use the same three letters in incompatible orders. Hence $|A_3 \cap A_4| = 0$.

Every intersection of three or four violation sets contains one of the incompatible pairs above, so all higher intersections are empty. Inclusion–exclusion gives

$$\begin{aligned} |U \setminus (A_1 \cup A_2 \cup A_3 \cup A_4)| &= P(26, 10) - 7P(22, 6) - 3(8P(23, 7)) \\ &\quad + 5P(20, 4) + 6P(21, 5) \\ &= 19,245,208,960,440. \end{aligned}$$

The coefficient 3 in the first subtraction groups three equal *single* counts. It does not assert that all intersections involving the same number of conditions have equal size: among the six pairwise intersections, two are nonzero and they have different sizes.

5.5 Euler’s totient function

Two positive integers are **relatively prime**, or **coprime**, if their greatest common divisor is 1. For a positive integer n , Euler’s **totient function** $\varphi(n)$ counts the integers in $\{1, 2, \dots, n\}$ that are relatively prime to n :

$$\varphi(n) = |\{a \in \{1, 2, \dots, n\} : \gcd(a, n) = 1\}|.$$

If we know the distinct prime divisors of n , inclusion–exclusion counts the integers that avoid divisibility by every one of them.

Theorem 8 (Product formula for Euler’s totient function). Let n be a positive integer with distinct prime divisors $p_1, p_2, \dots, p_m$. Then

$$\varphi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right).$$

For $n = 1$, the list of prime divisors is empty, the empty product is 1, and the formula gives $\varphi(1) = 1$.

Proof. Let $U = \{1, 2, \dots, n\}$. For each distinct prime divisor p_i of n , let A_i be the set of integers in U divisible by p_i . An integer in U is not relatively prime to n precisely when it belongs to at least one A_i . Thus $\varphi(n)$ counts the elements that belong to none of these sets.

For an index set $I \subseteq \{1, \dots, m\}$, an integer lies in $\bigcap_{i \in I} A_i$ precisely when it is divisible by $\prod_{i \in I} p_i$. The primes are distinct, and their product divides n , so exactly

$$\frac{n}{\prod_{i \in I} p_i}$$

elements of U satisfy these conditions. The complement form of inclusion–exclusion gives

$$\begin{aligned} \varphi(n) &= \sum_{I \subseteq \{1, \dots, m\}} (-1)^{|I|} \frac{n}{\prod_{i \in I} p_i} \\ &= n \prod_{i=1}^m \left(1 - \frac{1}{p_i}\right). \end{aligned}$$

The last equality is the expansion of the product: choosing $-1/p_i$ from the i th factor records the index i in I . □

Example 9 (Computing $\varphi(60)$). The distinct prime divisors of $60 = 2^2 \cdot 3 \cdot 5$ are 2, 3, and 5. The exponents in the prime factorization do not create additional violation sets: an integer fails to be coprime to 60 when it is divisible by 2, by 3, or by 5. Therefore

$$\varphi(60) = 60 \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{5}\right) = 60 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 16.$$

There are 16 integers in $\{1, 2, \dots, 60\}$ relatively prime to 60.

5.6 When only the number of imposed conditions matters

In the substring example, the number of strings in an intersection depends on which conditions we impose. Of the six pairwise intersections, four are empty and the other two have different cardinalities. We therefore had to examine the pairs one at a time.

Other problems have enough symmetry that the identities of the imposed conditions do not affect the count. In that situation, the sum over all subsets of conditions becomes a sum indexed only by the number of imposed conditions.

Theorem 10 (Inclusion–exclusion grouped by number of conditions). Let U be a finite set, and let A_i consist of the objects in U that satisfy condition i , for $1 \leq i \leq m$. Set $N_0 = |U|$. Suppose that for each

$k \in \{1, \dots, m\}$ there is a number N_k such that

$$\left| \bigcap_{i \in I} A_i \right| = N_k \quad \text{whenever } I \subseteq \{1, \dots, m\} \text{ and } |I| = k.$$

Then the number of objects satisfying at least one condition is

$$\sum_{k=1}^m (-1)^{k+1} C(m, k) N_k,$$

and the number satisfying none of the conditions is

$$\sum_{k=0}^m (-1)^k C(m, k) N_k.$$

Proof. For a fixed k , there are $C(m, k)$ index sets I of size k . By hypothesis, each corresponding intersection has size N_k . Therefore

$$\sum_{\substack{I \subseteq \{1, \dots, m\} \\ |I|=k}} \left| \bigcap_{i \in I} A_i \right| = C(m, k) N_k.$$

Substituting this equality into the general union formula gives the first formula. Substituting it into the complement formula gives the second. $\square$

The word *imposed* has a specific meaning here. An object counted by $\bigcap_{i \in I} A_i$ must satisfy every condition indexed by I , but it may also satisfy conditions whose indices are not in I . Thus N_k does not count objects satisfying *exactly* k conditions.

Example 11 (The same upper bound on five variables). How many integer solutions satisfy

$$x_1 + x_2 + x_3 + x_4 + x_5 = 30, \quad 0 \leq x_i \leq 9 \quad (1 \leq i \leq 5)?$$

Solution: Let U be the set of nonnegative solutions before the upper bounds are imposed. Stars and bars gives

$$N_0 = |U| = C(34, 4) = 46,376.$$

For each $i \in \{1, 2, 3, 4, 5\}$, let A_i be the violation set defined by $x_i \geq 10$.

Suppose that $I \subseteq \{1, 2, 3, 4, 5\}$ has size k . For every $i \in I$, subtract 10 from x_i . The transformed nonnegative variables have sum $30 - 10k$. The number of solutions depends on k , but not on which variables are named by I . Thus

k	0	1	2	3	4	5
N_k	$C(34, 4)$	$C(24, 4)$	$C(14, 4)$	$C(4, 4)$	0	0
	46,376	10,626	1,001	1	0	0

and the grouped complement formula gives

$$\begin{aligned} |U \setminus (A_1 \cup \dots \cup A_5)| &= C(5, 0)N_0 - C(5, 1)N_1 + C(5, 2)N_2 - C(5, 3)N_3 \\ &= 46,376 - 5(10,626) + 10(1,001) - 10(1) \\ &= 3,246. \end{aligned}$$

The calculation uses the same shifting method as [Example 6](#). In that example, the bounds on x_1, x_2, x_3 were 9, 7, and 4. The three single-intersection counts were different, as were the three pairwise-intersection counts, so we kept every term separate. Here all five variables have the same upper bound. Any chosen k violation conditions leave the same total $30 - 10k$, so the $C(5, k)$ intersections of size k all have cardinality N_k . The binomial coefficient is justified by this equality of cardinalities; it is not part of the general inclusion–exclusion formula.

The same argument yields a formula for any number of variables with a common upper bound.

Theorem 12 (Nonnegative solutions with a common upper bound). Let r, N, q be nonnegative integers with $r \geq 1$. The number of integer solutions to

$$x_1 + x_2 + \cdots + x_r = N, \quad 0 \leq x_i \leq q \quad (1 \leq i \leq r),$$

is

$$\sum_{k=0}^{\min\{r, \lfloor N/(q+1) \rfloor\}} (-1)^k C(r, k) C(N - k(q+1) + r - 1, r - 1).$$

Proof. Let U be the set of all nonnegative solutions to $x_1 + \cdots + x_r = N$. For each $i \in \{1, \dots, r\}$, let A_i contain the solutions with $x_i \geq q + 1$.

If $I \subseteq \{1, \dots, r\}$ has size k , subtracting $q + 1$ from every x_i with $i \in I$ gives a bijection from $\bigcap_{i \in I} A_i$ to the nonnegative solutions of

$$x'_1 + x'_2 + \cdots + x'_r = N - k(q + 1).$$

When the right side is nonnegative, stars and bars shows that the intersection has size

$$N_k = C(N - k(q + 1) + r - 1, r - 1).$$

When $N - k(q + 1) < 0$, the intersection is empty. This cardinality depends only on k , so the formula for satisfying none of the conditions in [Theorem 10](#) gives the stated sum. $\square$

Warning (Check symmetry before using $C(m, k)N_k$). The grouped formula requires every choice of k conditions to give the same intersection size. In [Example 7](#), some pairs merge into a longer block while other pairs are incompatible, so there is no single number N_2 . The general inclusion–exclusion formula still applies, but the pairwise intersections must be counted separately.

5.7 Graphs with no isolated vertices

A **simple graph** consists of a set of **vertices** together with a set of **edges**, where each edge is an unordered pair of distinct vertices. Thus a simple graph has no loops from a vertex to itself and no multiple edges joining the same pair. In this section the vertex set is

$$[n] = \{1, 2, \dots, n\},$$

so the vertices are labeled. Two graphs with different edge sets are counted as different graphs. A vertex is **isolated** if it belongs to no edge.

Problem 13 (Graphs without isolated vertices). How many simple graphs on the labeled vertex set $[n]$ have no isolated vertices? Derive a formula, and then evaluate it when $n = 5$.

We begin by counting all simple graphs on $[n]$, before excluding isolated vertices. There is one possible edge for each unordered pair of distinct vertices, so there are

$$C(n, 2) = \frac{n(n-1)}{2}$$

possible edges. For each pair we independently decide whether its edge is present or absent. The product rule therefore gives

$$2^{C(n,2)} = 2^{n(n-1)/2}$$

simple graphs on $[n]$.

For five vertices, the ten dashed segments in Figure 3 are the ten possible edges. A particular graph is obtained by retaining any subset of them. The drawing is not one graph being counted; it displays the ten independent edge choices used to count all graphs.

All possible edges on five labeled vertices

Each dashed segment is an independent include-or-omit choice.

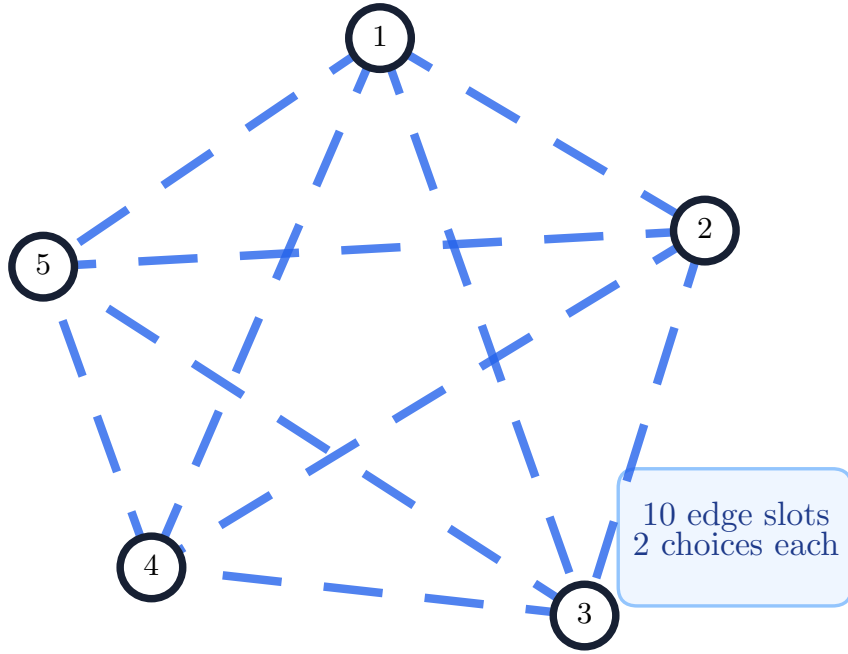


Figure 3: The $C(5, 2) = 10$ possible edges on five labeled vertices. Each edge may be present or absent independently, giving 2^{10} graphs.

Figure description: Five labeled vertices form a slightly irregular pentagon. A dashed blue segment joins every pair, displaying ten possible edges. Each segment represents an independent present-or-absent choice.

Let U be the set of all simple graphs on $[n]$. For each vertex $i \in [n]$, let

$$A_i = \{G \in U : i \text{ is isolated in } G\}.$$

A graph has no isolated vertices precisely when it belongs to none of the sets A_i .

Suppose $I \subseteq [n]$ and $|I| = k$. In a graph belonging to $\bigcap_{i \in I} A_i$, every edge having an endpoint in I must be absent. The only edges still available join two of the $n - k$ vertices outside I . There are $C(n - k, 2)$ such

pairs, and each corresponding edge may be present or absent independently. Consequently,

$$\left| \bigcap_{i \in I} A_i \right| = 2^{C(n-k, 2)}.$$

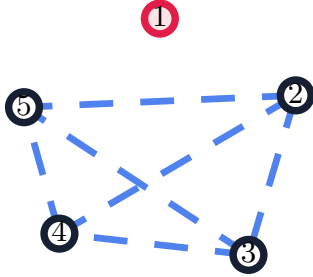
Here $C(r, 2) = 0$ when r is 0 or 1, because fewer than two vertices determine no possible edge. This number depends on k , but not on the identities of the vertices in I . The grouped complement form of inclusion–exclusion gives the formula

$$\sum_{k=0}^n (-1)^k C(n, k) 2^{C(n-k, 2)}. \quad (2)$$

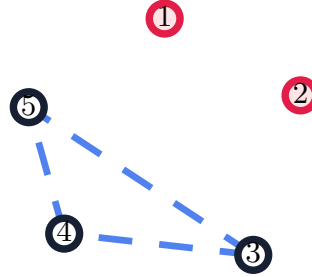
For $n = 5$, [Figure 4](#) shows a representative intersection for each value $1 \leq k \leq 5$. The red vertices are required to be isolated. Dashed edges are the choices that remain; an unshown edge is forced to be absent. The vertices not colored red are not required to be nonisolated. They may become isolated when some or all of their optional edges are omitted.

Named vertices required to be isolated

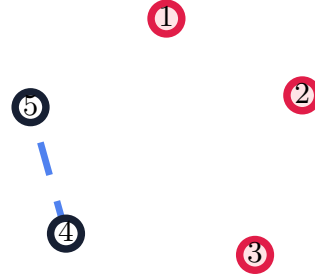
$I = 1$
6 optional edges;
 $N_1 = 2^6 = 64$



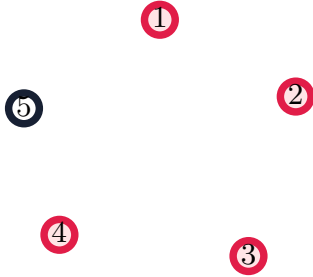
$I = 1, 2$
3 optional edges;
 $N_2 = 2^3 = 8$



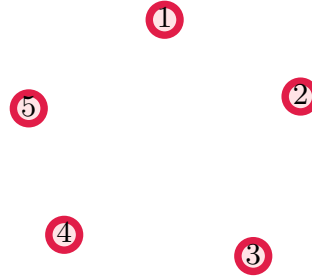
$I = 1, 2, 3$
1 optional edge;
 $N_3 = 2$



$I = 1, 2, 3, 4$
0 optional edges;
 $N_4 = 1$



$I = 1, 2, 3, 4, 5$
0 optional edges;
 $N_5 = 1$



How to read a panel

— — optional edge

i required isolated

Every unshown edge is forced absent.
Other vertices may also be isolated
when optional edges are omitted.

Figure 4: Representative intersections of k isolated-vertex conditions for graphs on $[5]$. After those conditions are imposed, only edges among the other $5 - k$ vertices remain optional.

Figure description: Five panels use the same slightly irregular pentagon of labeled vertices. In the panel for k isolated-vertex conditions, vertices 1 through k are red. Dashed blue edges join every pair of remaining vertices. For k equal to 1, 2, 3, 4, and 5, the panels show respectively 6, 3, 1, 0, and 0 optional edges, giving 64, 8, 2, 1, and 1 graphs. The complete inclusion–exclusion calculation is

$$\begin{aligned}
 & C(5, 0)2^{C(5, 2)} - C(5, 1)2^{C(4, 2)} + C(5, 2)2^{C(3, 2)} \\
 & \quad - C(5, 3)2^{C(2, 2)} + C(5, 4)2^{C(1, 2)} - C(5, 5)2^{C(0, 2)} \\
 & = 2^{10} - 5(2^6) + 10(2^3) - 10(2^1) + 5(2^0) - 2^0 \\
 & = 1024 - 320 + 80 - 20 + 5 - 1 \\
 & = 768.
 \end{aligned}$$

There are therefore 768 simple graphs on the labeled vertex set $[5]$ with no isolated vertices. Notice that the $k = 4$ and $k = 5$ intersections each contain one graph: the graph with no edges. If four vertices are required to be isolated, the fifth vertex cannot have an edge by itself and is isolated as well.

5.8 Derangements and fixed points

Let $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be a permutation of the first n positive integers. An index $i \in \{1, 2, \dots, n\}$ is a **fixed point** of π if $\pi(i) = i$.

Definition 14 (Derangement). A **derangement** of $\{1, 2, \dots, n\}$ is a permutation with no fixed points; that is, $\pi(i) \neq i$ for all $i \in \{1, 2, \dots, n\}$. The number of derangements of an n -element set is denoted by D_n or $!n$.

A classic illustration is the hat-check problem: if n people check their hats and the hats are returned completely at random, a derangement is an assignment in which no person receives their own hat.

Theorem 15 (Formula for derangements). For any integer $n \geq 1$, the number of derangements of $\{1, 2, \dots, n\}$ is

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!} = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} \right).$$

By convention, $D_0 = 1$.

Proof. Let U be the set of all $n!$ permutations of $\{1, 2, \dots, n\}$. For each $i \in \{1, \dots, n\}$, let A_i be the set of permutations that fix element i :

$$A_i = \{\pi \in U : \pi(i) = i\}.$$

A permutation is a derangement if and only if it belongs to none of the sets A_i , so $D_n = |U \setminus \bigcup_{i=1}^n A_i|$. Let $I \subseteq \{1, \dots, n\}$ with $|I| = k$. A permutation in the intersection $\bigcap_{i \in I} A_i$ fixes all k elements in I . The remaining $n - k$ elements may be permuted arbitrarily in $(n - k)!$ ways. Thus,

$$\left| \bigcap_{i \in I} A_i \right| = (n - k)!,$$

regardless of which k elements are chosen.

The number $(n - k)!$ depends only on the number k of fixed-point conditions, not on which elements are fixed. Therefore [Theorem 10](#) gives

$$D_n = \sum_{k=0}^n (-1)^k C(n, k) (n - k)!.$$

Using the identity $C(n, k)(n - k)! = \frac{n!}{k!(n - k)!} (n - k)! = \frac{n!}{k!}$, we factor out $n!$:

$$D_n = \sum_{k=0}^n (-1)^k \frac{n!}{k!} = n! \sum_{k=0}^n \frac{(-1)^k}{k!},$$

which proves the formula. □

The table below lists the first several values of D_n alongside $n!$:

n	0	1	2	3	4	5	6
$n!$	1	1	2	6	24	120	720
D_n	1	0	1	2	9	44	265

Example 16 (Returning items at random). Six students place their phones into a basket, and the phones are redistributed uniformly at random so that each student receives one phone.

1. In how many ways can the phones be returned so that no student receives their own phone?
2. In how many ways can the phones be returned so that exactly two students receive their own phones?

Solution:

1. This is the number of derangements of 6 items:

$$\begin{aligned}
 D_6 &= 6! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \right) \\
 &= 720 \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \right) \\
 &= 360 - 120 + 30 - 6 + 1 = 265.
 \end{aligned}$$

2. To have exactly two students receive their own phones, we first choose which two students receive their own phones in $C(6, 2)$ ways. The remaining 4 phones must then be deranged among the other 4 students so that no further student receives their own phone:

$$C(6, 2)D_4 = 15 \cdot 9 = 135.$$

The reasoning in part (2) of [Example 16](#) immediately generalizes to count permutations with any specified number of fixed points.

Corollary 17 (Permutations with exactly k fixed points). The number of permutations of $\{1, 2, \dots, n\}$ having exactly k fixed points (where $0 \leq k \leq n$) is

$$C(n, k)D_{n-k} = \frac{n!}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!}.$$

Proof. There are $C(n, k)$ ways to choose the k elements that remain fixed. For each such choice, the remaining $n - k$ elements must be deranged among themselves to ensure that no other element is fixed. There are D_{n-k} such derangements. By the product rule, the total number of permutations is $C(n, k)D_{n-k}$. $\square$

Remark 18 (Asymptotic probability of a derangement). The probability that a uniformly chosen permutation of n elements is a derangement is

$$\frac{D_n}{n!} = \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

As $n \rightarrow \infty$, this sum converges rapidly to the Taylor series for e^{-1} :

$$\lim_{n \rightarrow \infty} \frac{D_n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} \approx 0.367879.$$

Because the series alternates with terms decreasing monotonically to zero, $|D_n/n! - e^{-1}| \leq \frac{1}{(n+1)!}$. Multiplying by $n!$ shows that $|D_n - n!/e| \leq \frac{1}{n+1} \leq \frac{1}{2}$ for all $n \geq 1$. Consequently, D_n is the nearest

integer to $n!/e$ for every $n \geq 1$.

n	2	3	4	5	6	7	8
D_n	1	2	9	44	265	1854	14833
$n!/e$	0.736	2.207	8.829	44.146	264.871	1854.112	14832.899

5.9 Counting surjective functions and full-alphabet strings

Let X and Y be finite sets with $|X| = n$ and $|Y| = m$. A function $f : X \rightarrow Y$ is **surjective**, or **onto**, if every element of Y is in the image of f : for each $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.

We may record a function $f : X \rightarrow Y$ as a string of length n after choosing an order for the elements of X . The character in the position belonging to x is $f(x)$. Under this correspondence, surjectivity means that the string contains every symbol of the alphabet Y at least once.

The condition that a function omit a specified output is easier to count than surjectivity itself. If k specified elements of Y are omitted, each input has $m - k$ possible images. This count depends only on the number k of omitted outputs, so [Theorem 10](#) applies.

Theorem 19 (Number of surjective functions). Let X and Y be finite sets with $|X| = n$ and $|Y| = m$, where $n, m \geq 1$. The number of surjective functions $f : X \rightarrow Y$ is

$$\sum_{k=0}^m (-1)^k C(m, k) (m - k)^n.$$

Equivalently, this is the number of length- n strings over an m -symbol alphabet that use every symbol.

Proof. Let U be the set of all m^n functions from X to Y . For each $y \in Y$, let A_y be the set of functions whose image omits y :

$$A_y = \{f \in U : y \notin f(X)\}.$$

A function is surjective if and only if it belongs to none of the sets A_y .

For any subset $K \subseteq Y$ of size k , the intersection $\bigcap_{y \in K} A_y$ consists of functions whose image is contained in $Y \setminus K$. There are $(m - k)^n$ such functions, regardless of which k elements form K . The formula for satisfying none of the m omission conditions in [Theorem 10](#) gives the result. $\square$

Example 20 (Surjective functions from a 5-set to a 3-set). How many surjective functions are there from a set of 5 elements to a set of 3 elements?

Solution: Setting $n = 5$ and $m = 3$ in [Theorem 19](#) gives

$$\begin{aligned} \sum_{k=0}^3 (-1)^k C(3, k) (3 - k)^5 &= C(3, 0) 3^5 - C(3, 1) 2^5 + C(3, 2) 1^5 - C(3, 3) 0^5 \\ &= 243 - 3 \cdot 32 + 3 \cdot 1 - 0 \\ &= 150. \end{aligned}$$

Thus there are 150 functions whose image contains all three target elements. The same number counts length-5 strings over a three-symbol alphabet in which every symbol occurs.

5.9.1 Connection with Stirling numbers

The **Stirling number of the second kind** $S(n, m)$ is the number of ways to partition an n -element set into m nonempty, unlabeled subsets. Every surjective function $f : X \rightarrow Y$ produces such a partition: its m parts are the nonempty preimages $f^{-1}(y)$ for $y \in Y$. Conversely, after choosing a partition into m parts, we may bijectively assign those parts to the m elements of Y in $m!$ ways. Therefore

$$m!S(n, m) = \sum_{k=0}^m (-1)^k C(m, k)(m - k)^n,$$

and hence

$$S(n, m) = \frac{1}{m!} \sum_{k=0}^m (-1)^k C(m, k)(m - k)^n.$$

When $n < m$, the pigeonhole principle says that no function $X \rightarrow Y$ can be surjective. Consequently $S(n, m) = 0$, and the alternating sum also equals 0.

5.10 Summary of the inclusion–exclusion framework

To apply the principle of inclusion–exclusion systematically to an enumeration problem:

1. **Define the universe U :** Identify the ambient set of all possible configurations without the restrictive conditions, and verify that $|U|$ can be computed.
2. **Define the violation sets A_i :** Formulate each restrictive condition so that failing condition i corresponds to membership in $A_i \subseteq U$.
3. **Characterize the intersections:** For an arbitrary index subset $I \subseteq \{1, \dots, m\}$, determine what mathematical structure an object in $\bigcap_{i \in I} A_i$ possesses.
4. **Count the intersections:** Compute $|\bigcap_{i \in I} A_i|$. If the answer is N_k for every index set I of size k , group the $C(m, k)$ equal terms. Check this hypothesis for each k ; do not infer it from the one-condition counts alone.
5. **Evaluate the alternating sum:** Substitute the intersection counts into the union formula or the complement formula, paying attention to the alternating signs.

The applications differ mainly in how an imposed set of conditions changes an object. Upper-bound violations reserve units before stars and bars; compatible substring conditions glue letters into blocks while incompatible ones give empty intersections; divisibility conditions force multiples of selected primes; isolated-vertex conditions remove every edge incident to the named vertices; fixed-point conditions remove elements from the part of a permutation still free to move; and omitted-output conditions reduce the available alphabet. Inclusion–exclusion supplies the alternating signs after these intersections have been counted.

5.11 Further reading

The [course resources page](#) links to the reference texts for this course.

- *Applied Combinatorics* by Tucker, Sections 7.1–7.4, provides additional examples of inclusion–exclusion, rook polynomials, and general derangement problems.
- *Discrete Mathematics: An Open Introduction* by Levin, Sections 3.3 and 3.8, covers Venn diagram bookkeeping, onto functions, and counting multisets with bounded multiplicities.
- *Counting Rocks!* by Schepers, Section 2.7, develops the algebraic indicator function approach.