

4. Binomial expansions, Pascal paths, and Catalan numbers

Combinatorics

Fall 2026

The preceding notes developed combinations, multinomial coefficients, lattice paths, and stars and bars as counting models. A new problem rarely announces which model it needs, so we begin by comparing four similar-looking choices whose outcomes are very different.

Then, we will see how the coefficients $C(n, k)$ appear in many surprising circumstances: in expansions of polynomials, in Pascal's triangle, and recursive lattice-path counts. Finally, imposing restrictions on those paths leads to hard counting problem. A famous one is that of the Catalan numbers and the reflection principle that counts them. These numbers also appear in counting ways to parenthesise expressions. After studying these notes, you will be able to do the following.

- Decide which kind of choice describes a problem by determining whether order matters and whether repetition is allowed.
- Model combinations with repetition as multiplicity vectors and count them.
- Explain the relationship among stars and bars, cumulative totals, weakly increasing sequences, and strictly increasing sequences.
- Translate lower bounds for integer solutions so that stars and bars applies.
- Use the binomial and multinomial theorems to extract coefficients from powers of polynomials.
- Verify Pascal's recurrence algebraically and prove it combinatorially.
- Compute lattice-path counts recursively and count paths that pass through or avoid a specified point.
- Use tail reflection to derive the Catalan-number formula and relate Catalan paths to balanced-parenthesis strings.

4.1 A bakery problem: four kinds of choice

A bakery sells ten types of doughnuts. A manager wants to buy six doughnuts for six employees. Four different purchasing scenarios will help us compare the available models.

Problem 1 (Buying doughnuts for six employees). Which standard kind of choice describes each scenario?

1. The manager decides which type of doughnut to buy for each employee and may choose the same type for several employees.
2. The manager again chooses one type for each employee, but decides not to buy the same type twice.
3. The manager selects six different types and lets the employees decide who receives which one.
4. The manager buys six doughnuts to bring to the employees. Several doughnuts may have the same type, and the manager does not assign them to particular employees.

4.1.1 Modeling the four scenarios

The important question is not only whether the manager purchases six doughnuts. We must ask what information one outcome retains.

1. In the first scenario, the employees play the role of six labeled positions. For each employee, the manager chooses one of ten types and may repeat the same type. An outcome is a length-6 string of doughnut types, or equivalently a function from the employees to the types. The number of choices is 10^6 .
2. The employees are still labeled, so assigning glazed to Alice and chocolate to Bob is different from assigning chocolate to Alice and glazed to Bob. Repetition is now forbidden. This is a 6-permutation of the ten types, and the number of choices is $P(10, 6) = \frac{10!}{4!}$.
3. The manager selects the six types before any employee receives one. Order and employee assignments are not part of the purchase, and the manager repeats no type. This is an ordinary combination, with $C(10, 6)$ possible selections.
4. Order and employee assignments are again irrelevant, but repetition is allowed. The manager might buy six glazed doughnuts, six different types, or anything in between. This is a **combination with repetition**, and the number of choices is $C(10 + 6 - 1, 6) = C(15, 6)$.

The four scenarios occupy the four rows of the familiar choice table:

Order matters?	Repetition allowed?	Model	Count
yes	yes	strings or functions	n^k
yes	no	permutations	$P(n, k)$
no	no	combinations	$C(n, k)$
no	yes	combinations with repetition	$C(n + k - 1, k)$

In this table, n is the number of available types and k is the number of objects selected. The bakery story shows why the wording of the problem is not enough by itself: the mathematical model depends on what counts as a different outcome.

4.2 Combinations with repetition and cumulative totals

Let us return to the fourth bakery scenario. If x_i is the number of doughnuts of type i , then a nonnegative integer solution describes a purchase:

$$x_1 + x_2 + \cdots + x_{10} = 6.$$

More generally, selecting k objects from n types with repetition is the same problem as finding the nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_n = k.$$

4.2.1 The separator count

The stars-and-bars construction places k stars in a row and uses $n - 1$ separators to divide them among the n types. There are $k + n - 1$ positions altogether. Choosing the separator positions gives

$$C(k + n - 1, n - 1)$$

solutions. Choosing the star positions instead gives the equal expression

$$C(k + n - 1, k).$$

Thus the six-doughnut purchase in the fourth scenario can be made in

$$C(15, 9) = C(15, 6) = 5005$$

ways.

Can we explain this count using another combinatorial encoding? Yes. We can record the separator positions as cumulative totals.

4.2.2 Cumulative totals and weakly increasing sequences

A sequence is **weakly increasing** if each term is at least the preceding term:

$$a_1 \leq a_2 \leq \cdots \leq a_r.$$

It is **strictly increasing** if each inequality is strict:

$$a_1 < a_2 < \cdots < a_r.$$

Repeated values are allowed in a weakly increasing sequence but not in a strictly increasing sequence. For a nonnegative solution of $x_1 + \cdots + x_n = k$, we define

$$\begin{aligned} y_1 &= x_1, \\ y_2 &= x_1 + x_2, \\ &\vdots \\ y_{n-1} &= x_1 + x_2 + \cdots + x_{n-1}, \\ y_n &= x_1 + x_2 + \cdots + x_{n-1} + x_n \end{aligned}$$

Since every x_i is nonnegative,

$$0 \leq y_1 \leq y_2 \leq \cdots \leq y_{n-1} \leq y_n = k.$$

The number y_i records how many stars appear before separator i . Two separators may have the same number of stars before them, which is why the sequence is only weakly increasing. The final total $y_n = k$ is forced, so we omit it from the encoding.

This construction is reversible. If numbers satisfy $0 \leq y_1 \leq \cdots \leq y_{n-1} \leq k$, then we recover

$$\begin{aligned} x_1 &= y_1, \\ x_i &= y_i - y_{i-1} \quad (2 \leq i \leq n-1), \\ x_n &= k - y_{n-1}. \end{aligned}$$

Thus solutions of $x_1 + \cdots + x_n = k$ are in bijection with weakly increasing sequences of length $n-1$ selected from

$$\{0, 1, \dots, k\}.$$

Such a sequence is a choice with repetition of $n-1$ elements from $k+1$ possible values. Applying the combination-with-repetition formula with these roles exchanged gives

$$C((k+1) + (n-1) - 1, n-1) = C(k+n-1, n-1) = C(k+n-1, k),$$

exactly as in the separator count.

Example 2 (Encoding the separators by cumulative totals). The solution $(x_1, x_2, x_3, x_4) = (2, 1, 2, 2)$ of $x_1 + x_2 + x_3 + x_4 = 7$ gives

$$(y_1, y_2, y_3) = (2, 3, 5).$$

In a stars-and-bars string, the three separators occur after 2, 3, and 5 stars. Conversely, the differences

$$2, \quad 3 - 2, \quad 5 - 3, \quad 7 - 5$$

recover $(2, 1, 2, 2)$.

4.2.3 An alternative proof of the combinations with repetitions formula

We already know that the number of nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_n = k$$

is $C(k + n - 1, n - 1)$. We obtained this answer by arranging k stars and $n - 1$ separators. Let us now obtain the same answer in a different way, using an ordinary combination.

An ordinary combination is a choice of distinct elements from a set. Thus our plan is to encode each solution as a subset. The cumulative totals from the preceding section almost do this, but they are only weakly increasing: two consecutive totals can be equal when some x_i is zero. To make the totals strictly increasing, we first make every part positive.

Theorem 3 (Combinations with repetition, revisited). The number of nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_n = k$$

is

$$C(k + n - 1, n - 1) = C(k + n - 1, k).$$

Equivalently, this is the number of ways to select k objects from n types when repetition is allowed.

Proof. We start with a nonnegative solution $(x_1, \dots, x_n)$ and define

$$z_i = x_i + 1 \quad (1 \leq i \leq n).$$

We have added one to each of the n variables, so

$$z_1 + z_2 + \cdots + z_n = k + n.$$

More importantly, every z_i is positive. This guarantees that the partial sums increase at every step. Next, we record the first $n - 1$ partial sums:

$$\begin{aligned} s_1 &= z_1, \\ s_2 &= z_1 + z_2, \\ &\vdots \\ s_{n-1} &= z_1 + z_2 + \cdots + z_{n-1}. \end{aligned}$$

Since each z_i is at least 1, these numbers satisfy

$$1 \leq s_1 < s_2 < \cdots < s_{n-1} \leq k + n - 1.$$

The final inequality holds because at least 1 must remain for z_n . We have therefore obtained an ordinary $(n - 1)$ -element subset

$$\{s_1, s_2, \dots, s_{n-1}\} \subseteq \{1, 2, \dots, k + n - 1\}.$$

We must check that this encoding does not lose any information. For any $(n - 1)$ -element subset of $\{1, 2, \dots, k + n - 1\}$, we write its elements in increasing order:

$$1 \leq s_1 < s_2 < \cdots < s_{n-1} \leq k + n - 1.$$

We recover the positive integers $z_1, \dots, z_n$ by taking consecutive differences:

$$z_1 = s_1, \quad z_i = s_i - s_{i-1} \quad (2 \leq i \leq n-1), \quad z_n = k + n - s_{n-1}.$$

Every one of these numbers is positive. Moreover, their sum telescopes:

$$s_1 + (s_2 - s_1) + \dots + (s_{n-1} - s_{n-2}) + (k + n - s_{n-1}) = k + n.$$

Finally, we define $x_i = z_i - 1$. The resulting x_i are nonnegative, and their sum is $(k + n) - n = k$, so they form a solution of the original equation.

The two constructions undo one another. We have therefore established a bijection between the solutions of $x_1 + \dots + x_n = k$ and the $(n-1)$ -element subsets of $\{1, 2, \dots, k + n - 1\}$. The number of such subsets is

$$C(k + n - 1, n - 1) = C(k + n - 1, k).$$

□

Example 4 (Following one solution through the bijection). Let us follow the earlier solution

$$(x_1, x_2, x_3, x_4) = (2, 1, 2, 2)$$

of $x_1 + x_2 + x_3 + x_4 = 7$. Adding 1 to each entry gives

$$(z_1, z_2, z_3, z_4) = (3, 2, 3, 3),$$

whose sum is $7 + 4 = 11$. The first three partial sums are

$$(s_1, s_2, s_3) = (3, 5, 8).$$

Thus the solution corresponds to the three-element subset $\{3, 5, 8\}$ of $\{1, 2, \dots, 10\}$.

Starting with $\{3, 5, 8\}$, consecutive differences recover

$$3, \quad 5 - 3, \quad 8 - 5, \quad 11 - 8,$$

which gives $(3, 2, 3, 3)$. Subtracting 1 from each entry returns the original solution $(2, 1, 2, 2)$.

Remark 5 (From weakly increasing to strictly increasing). There is a direct connection between this proof and the weakly increasing sequence from the preceding section. We had

$$y_i = x_1 + x_2 + \dots + x_i.$$

Since $z_j = x_j + 1$, the new partial sums satisfy

$$s_i = z_1 + \dots + z_i = y_i + i.$$

Thus we turn a weakly increasing sequence into a strictly increasing one by adding 1 to its first term, 2 to its second term, and so on:

$$(y_1, y_2, \dots, y_{n-1}) \mapsto (y_1 + 1, y_2 + 2, \dots, y_{n-1} + n - 1).$$

Subtracting i from the i th term reverses the transformation. Therefore

$$0 \leq y_1 \leq \dots \leq y_{n-1} \leq k$$

is equivalent to

$$1 \leq s_1 < \dots < s_{n-1} \leq k + n - 1.$$

4.2.4 Solutions with lower bounds

Stars and bars applies directly when every variable may be any nonnegative integer. If a variable must instead satisfy $x_i \geq \ell_i$, reserve its required amount ℓ_i first. The part still available for distribution is then counted with nonnegative variables.

Theorem 6 (Integer solutions with lower bounds). For integers $N, \ell_1, \dots, \ell_n$, the number of integer solutions to

$$x_1 + x_2 + \dots + x_n = N, \quad x_i \geq \ell_i$$

is

$$C\left(N - \sum_{i=1}^n \ell_i + n - 1, n - 1\right)$$

when $N - \sum_i \ell_i \geq 0$. If that difference is negative, there are no solutions.

Proof. We set $u_i = x_i - \ell_i$. Then $u_i \geq 0$, and the original equation becomes

$$u_1 + u_2 + \dots + u_n = N - (\ell_1 + \dots + \ell_n).$$

The substitution is reversible: from a solution in the u_i , recover $x_i = u_i + \ell_i$. Stars and bars applied to the transformed equation gives the formula. $\square$

Example 7 (Asymmetric lower bounds). How many integer solutions satisfy

$$x_1 + x_2 + x_3 + x_4 = 30, \quad x_1 \geq 2, \quad x_2 \geq 0, \quad x_3 \geq 5, \quad x_4 \geq 1.$$

Solution: We define

$$u_1 = x_1 - 2, \quad u_2 = x_2, \quad u_3 = x_3 - 5, \quad u_4 = x_4 - 1.$$

The new variables are nonnegative and satisfy

$$u_1 + u_2 + u_3 + u_4 = 30 - (2 + 0 + 5 + 1) = 22.$$

Therefore the number of solutions is

$$C(22 + 4 - 1, 4 - 1) = C(25, 3) = 2300.$$

Warning (Lower bounds and upper bounds behave differently). The method we saw allows us to map an integer solution problem with a lower bound (for example $x_1 > 10$) to a problem of counting non-negative integer solutions. This method does not work if we have upper bound conditions, for example $x_1 \leq 10$.

We can still solve the problem when there is an upper bound condition on only one variable, by using the subtraction principle. However, with several upper bounds, such as $x_1 \leq 10$ and $x_2 \leq 7$ will require us to use the inclusion–exclusion principle.

4.2.5 Summary

- The number of nonnegative integer solutions of

$$x_1 + x_2 + \dots + x_n = k$$

is

$$C(k + n - 1, n - 1) = C(k + n - 1, k).$$

- If the variables have lower bounds $x_i \geq \ell_i$, we subtract the lower bounds first. The remaining total is $N - \sum_i \ell_i$.
- Cumulative totals encode a solution as a weakly increasing sequence. Adding i to its i th term converts it into a strictly increasing sequence, which is the same as an ordinary subset.

4.3 The binomial theorem

Let us consider the product

$$(x + y)^n = \underbrace{(x + y)(x + y) \cdots (x + y)}_{n \text{ factors}}.$$

We obtain every raw term in its expansion by choosing either x or y from each factor. To obtain $x^k y^{n-k}$, exactly k of the factors must contribute x ; all other factors contribute y . The coefficient is therefore the number of ways to choose those k factors.

Theorem 8 (Binomial theorem). For every nonnegative integer n ,

$$(x + y)^n = \sum_{k=0}^n C(n, k) x^k y^{n-k}.$$

Proof. Label the n factors by their positions. The subset of positions from which we choose x determines a term. If that subset has size k , the resulting term is $x^k y^{n-k}$. There are $C(n, k)$ such subsets. Grouping the raw terms by k gives the displayed expansion. $\square$

For example, row 5 of the binomial coefficients gives

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

The same reasoning works when the two terms have numerical weights.

Example 9 (Extracting a weighted coefficient). What is the coefficient of x^4 in $(2x - 3)^7$?

Solution: To obtain x^4 , we choose $2x$ from exactly four of the seven factors and choose -3 from the remaining three. The coefficient is

$$C(7, 4)2^4(-3)^3 = -15120.$$

The binomial coefficient chooses the contributing factors; the powers 2^4 and $(-3)^3$ record the numerical contribution from those choices.

Example 10 (Accounting for an outside power). What is the coefficient of x^9 in $x^3(1 + 2x)^8$?

Solution: The outside factor already contributes x^3 , so we need x^6 from $(1 + 2x)^8$. We choose $2x$ from six factors. The coefficient is

$$C(8, 6)2^6 = 1792.$$

Two useful identities follow immediately from the binomial theorem. Setting $x = y = 1$ gives

$$\sum_{k=0}^n C(n, k) = 2^n,$$

which also counts all subsets of an n -element set by their sizes. For $n \geq 1$, setting $x = 1$ and $y = -1$ gives

$$\sum_{k=0}^n (-1)^{n-k} C(n, k) = 0.$$

Equivalently, the sum of the even-indexed entries in row n equals the sum of the odd-indexed entries.

4.4 The multinomial theorem

When every factor offers r choices rather than two, a multinomial coefficient counts the multiplicities of the choices.

Theorem 11 (Multinomial theorem). For every integer $r \geq 1$ and every nonnegative integer n ,

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1 + \cdots + k_r = n} \frac{n!}{k_1! k_2! \cdots k_r!} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r},$$

where the sum ranges over all nonnegative integer solutions of $k_1 + \cdots + k_r = n$.

Proof. We choose one term from each of the n labeled factors. A raw term equals $x_1^{k_1} \cdots x_r^{k_r}$ exactly when the resulting choice string contains k_i copies of x_i for every i . The number of such strings is

$$C(n; k_1, \dots, k_r) = \frac{n!}{k_1! \cdots k_r!}.$$

Grouping equal raw terms proves the formula. □

Example 12 (A multinomial coefficient with weights). What is the coefficient of $x^2 y^3 z$ in $(2x - y + 3z)^6$?

Solution: The exponents sum to 6, so the requested term can occur. We choose the two factors contributing $2x$, the three contributing $-y$, and the one contributing $3z$. The coefficient is

$$\frac{6!}{2!3!1!} 2^2 (-1)^3 3 = -720.$$

Before we compute a multinomial coefficient, we check the total degree. For example, the coefficient of $a^2 b^2 c^3$ in $(a + b + c)^6$ is zero because the requested exponents sum to 7, while every term in the expansion has total degree 6.

Remark 13 (How many distinct monomials appear?). A nonnegative solution of $k_1 + \cdots + k_r = n$ determines a monomial in $(x_1 + \cdots + x_r)^n$. Stars and bars shows that the number of distinct monomials is $C(n + r - 1, r - 1)$.

4.5 Pascal's recurrence and Pascal's triangle

The factorial formula is not the only way to compute binomial coefficients. A local counting argument relates each interior entry to two entries in the preceding row.

Theorem 14 (Pascal's recurrence). If $n \geq 2$ and $1 \leq k \leq n - 1$, then

$$C(n, k) = C(n - 1, k) + C(n - 1, k - 1).$$

The boundary values are $C(n, 0) = C(n, n) = 1$.

Algebraic verification. Using the factorial formula and a common denominator,

$$\begin{aligned}
 C(n-1, k) + C(n-1, k-1) &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
 &= \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!k}{k!(n-k)!} \\
 &= \frac{(n-1)!((n-k) + k)}{k!(n-k)!} \\
 &= \frac{n!}{k!(n-k)!} \\
 &= C(n, k).
 \end{aligned}$$

□

Combinatorial proof. For an n -element set S with a distinguished element $s \in S$, we partition the k -element subsets of S into two classes.

If a subset does not contain s , we choose all k elements from the remaining $n-1$ elements, giving $C(n-1, k)$ possibilities. If it contains s , we choose its other $k-1$ elements from the remaining $n-1$, giving $C(n-1, k-1)$ possibilities. The two classes are disjoint and exhaustive, so the sum rule gives the recurrence. □

Starting with the boundary entries 1 and using the recurrence produces Pascal's triangle:

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

We obtain entry $C(n, k)$ by adding the two entries immediately above it. The symmetry $C(n, k) = C(n, n-k)$ is visible in the triangle and corresponds to choosing the included elements or, equivalently, the omitted elements.

4.6 Recursive lattice-path counts

For nonnegative integers a and b , we write $L(a, b)$ for the number of northeast lattice paths from $(0, 0)$ to (a, b) . If $a, b > 0$, a path reaches (a, b) either from $(a-1, b)$ by a right step or from $(a, b-1)$ by an up step. These cases are disjoint, so

$$L(a, b) = L(a-1, b) + L(a, b-1).$$

Along the axes there is only one path, so $L(a, 0) = L(0, b) = 1$. These are the same recurrence and boundary data as Pascal's triangle. Therefore

$$L(a, b) = C(a+b, a) = C(a+b, b),$$

in agreement with the step-string argument from the preceding notes.

We can use the recurrence to fill an entire grid. We begin with 1 along the coordinate axes. At every interior vertex, we add the labels at the vertex to its left and the vertex below it.

Figure 1 shows this computation near the point $(4, 3)$.

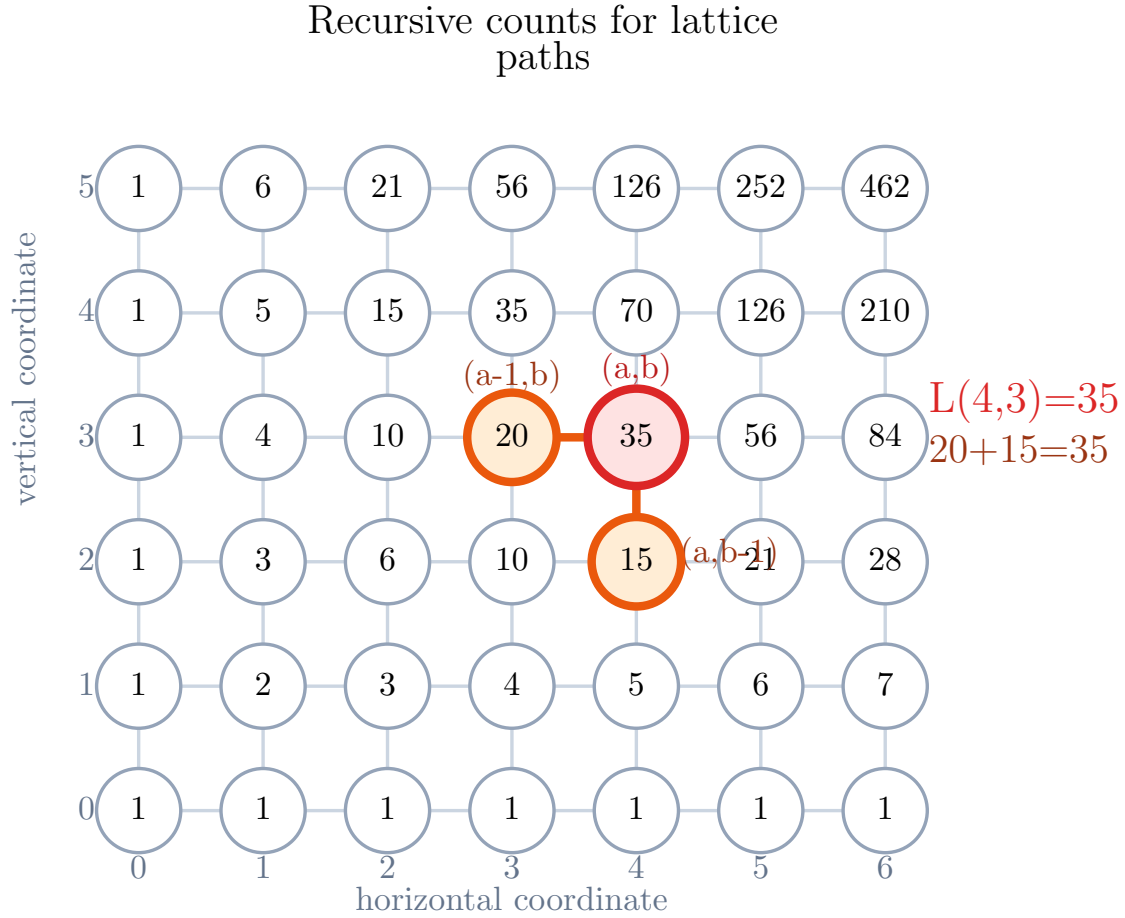


Figure 1: A recursive table of northeast-path counts. The highlighted point $(a, b) = (4, 3)$ can be reached from $(3, 3)$ in 20 ways and from $(4, 2)$ in 15 ways, so its label is $20 + 15 = 35$.

Figure description: A seven-column by six-row lattice whose vertices are labeled by path counts. The vertex $(4, 3)$, labeled 35, is highlighted in red. Its left neighbor $(3, 3)$, labeled 20, and lower neighbor $(4, 2)$, labeled 15, are highlighted in orange, with arrows pointing to $(4, 3)$.

This table is a rotated portion of Pascal's triangle. It also suggests a way to handle forbidden vertices when no direct formula is available: we assign a forbidden vertex the value zero, then continue filling the table by the same left-plus-below rule.

4.6.1 Paths through a specified point

For $0 \leq c \leq a$ and $0 \leq d \leq b$, a path from $(0, 0)$ to (a, b) that passes through (c, d) decomposes uniquely into a path from $(0, 0)$ to (c, d) followed by a path from (c, d) to (a, b) . The product rule gives

$$C(c + d, c)C(a + b - c - d, a - c).$$

The total number of paths to (a, b) is $C(a + b, a)$, so the number avoiding (c, d) is

$$C(a + b, a) - C(c + d, c)C(a + b - c - d, a - c).$$

Example 15 (Passing through or avoiding $(4, 7)$). A path from $(0, 0)$ to $(10, 10)$ that passes through $(4, 7)$ consists of a path with four right and seven up steps, followed by a path with six right and three up steps. Thus the number passing through the point is

$$C(11, 4)C(9, 6) = 330 \cdot 84 = 27720.$$

There are $C(20, 10) = 184756$ paths altogether, so the number avoiding $(4, 7)$ is

$$C(20, 10) - C(11, 4)C(9, 6) = 157036.$$

Warning 16 (Feasibility of the specified point). The product formula assumes that (c, d) is coordinatewise between the start and endpoint. A northeast path from $(0, 0)$ to (a, b) cannot pass through a point with a negative coordinate, an x -coordinate greater than a , or a y -coordinate greater than b .

4.7 Catalan paths and the reflection principle

There are $C(2n, n)$ northeast paths from $(0, 0)$ to (n, n) . We now impose a global restriction: at every point of the path, require

$$y \leq x.$$

In other words, the path may touch the diagonal $y = x$, but it may never go above it. A path satisfying this restriction is called a **Catalan path** or a **Dyck path**.

Definition 17 (Catalan numbers). The n th **Catalan number** C_n is the number of northeast lattice paths from $(0, 0)$ to (n, n) that never go above the line $y = x$. We include the empty path, so $C_0 = 1$.

For $n = 3$, [Figure 2](#) displays all five paths that obey the diagonal restriction.

The five Catalan paths from $(0,0)$ to $(3,3)$

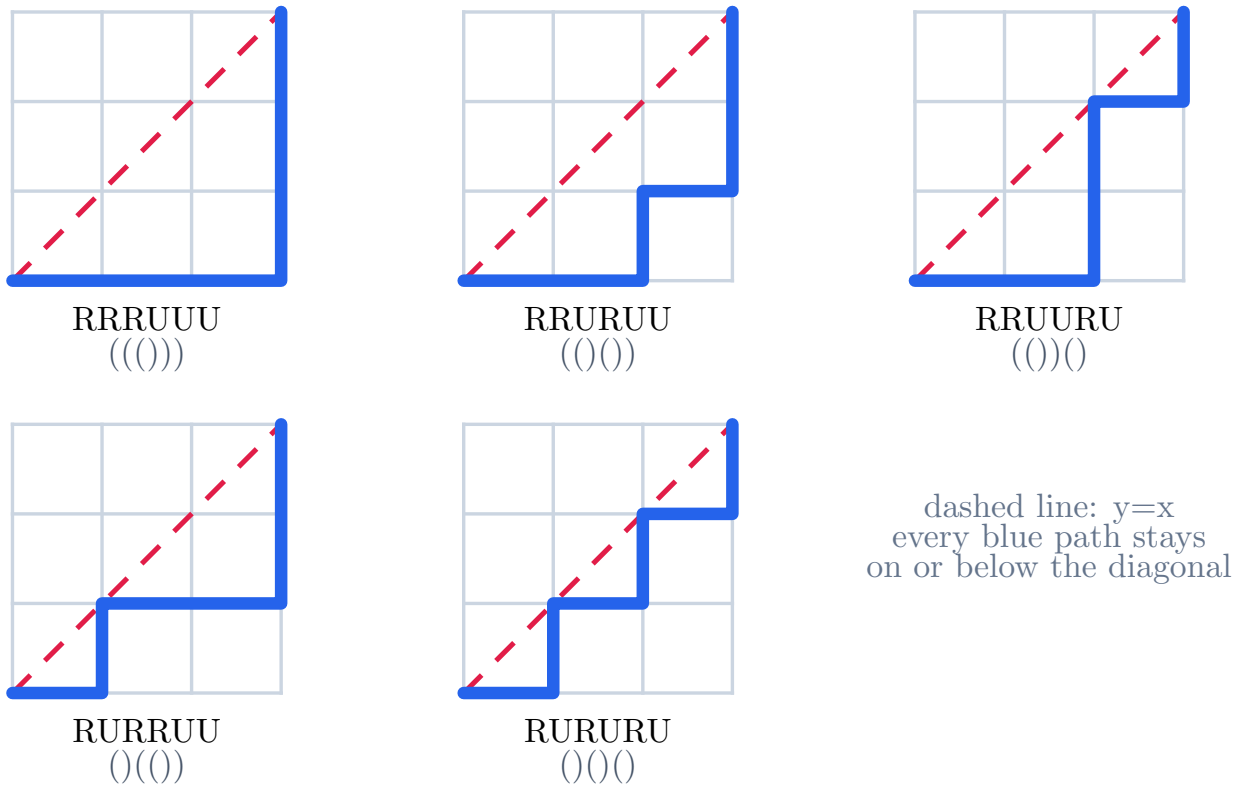


Figure 2: The five Catalan paths for $n = 3$. The corresponding step strings and balanced-parenthesis strings are displayed below the paths.

Figure description: Five separate three-by-three grids show all northeast paths from $(0,0)$ to $(3,3)$ that stay on or below the diagonal. The paths are labeled $RRRUUU$, $RRURUU$, $RRUURU$, $RURRUU$, and $RURURU$, together with their balanced-parenthesis strings.

To count the valid paths, we subtract the paths that cross the diagonal. The reflection principle gives a bijective count of those bad paths.

[Figure 3](#) previews the transformation: reflection leaves the path unchanged until its first crossing and reflects the remaining tail.

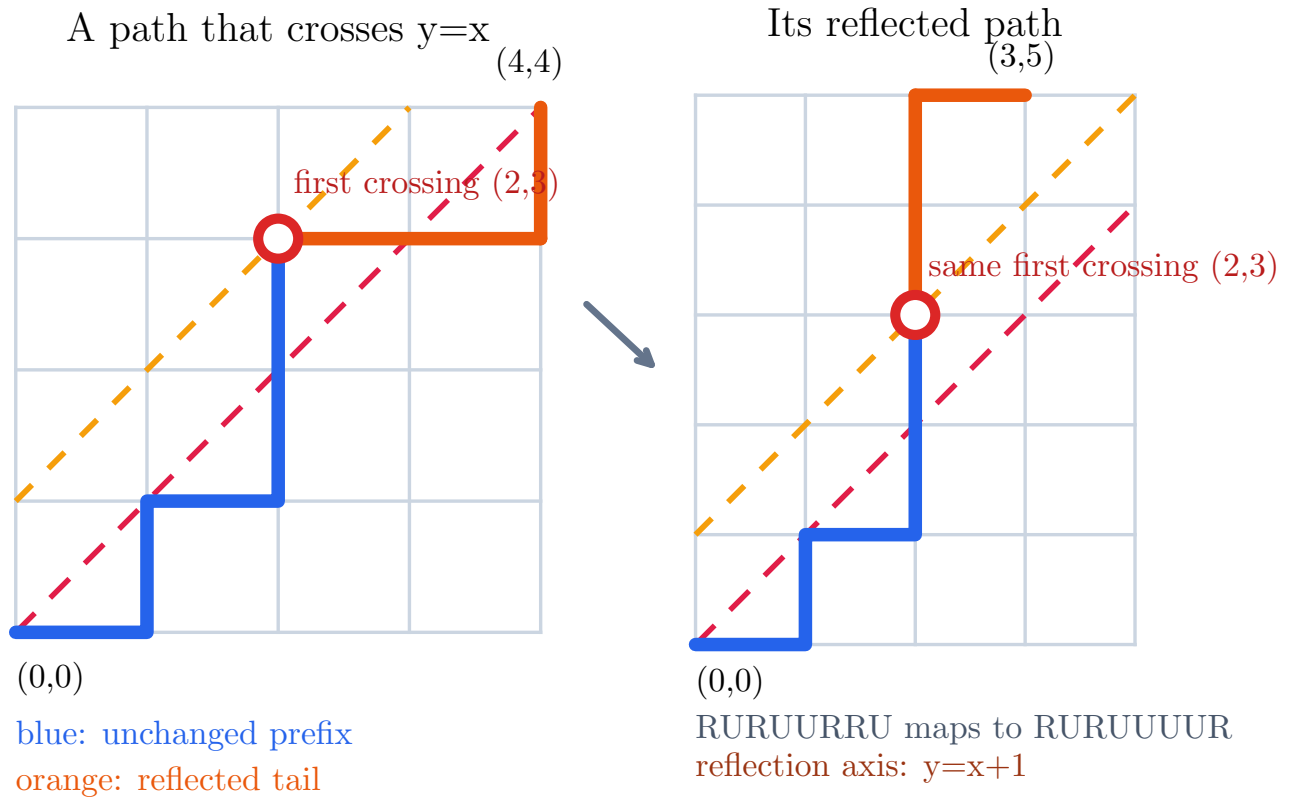


Figure 3: Reflecting the tail after the first crossing changes a bad path to $(4,4)$ into a path to $(3,5)$. The construction is reversible.

Figure description: Two grids compare a bad path from $(0,0)$ to $(4,4)$ with a path from $(0,0)$ to $(3,5)$. Both paths have the same blue prefix ending at the first point $(2,3)$ on the line y equals x plus 1. The orange tail of the first path is reflected across that line to produce the orange tail of the second path.

Theorem 18 (Catalan formula). For every nonnegative integer n ,

$$C_n = C(2n, n) - C(2n, n-1) = \frac{1}{n+1} C(2n, n).$$

Proof. For $n = 0$, the one path is empty, and both displayed expressions equal 1. Now assume $n \geq 1$. A bad path has a first point at which it reaches the line $y = x + 1$. We keep the path unchanged up to that point and reflect the *tail*—the portion from the first crossing to the endpoint—across the line $y = x + 1$. The crossing point lies on the reflecting line and therefore stays fixed. On the tail, reflection exchanges right steps and up steps.

Reflection across $y = x + 1$ sends a point (x, y) to $(y - 1, x + 1)$. In particular, it sends the original endpoint (n, n) to $(n - 1, n + 1)$. The resulting path therefore goes from $(0, 0)$ to $(n - 1, n + 1)$.

This map is reversible. Any northeast path from $(0, 0)$ to $(n - 1, n + 1)$ must reach the line $y = x + 1$. We keep its prefix through the first point on that line and reflect its tail across the line. This produces a bad path to (n, n) , and reflecting the same tail again recovers the original path. Thus bad paths to (n, n) are in bijection with all paths to $(n - 1, n + 1)$. There are

$$C(2n, n - 1)$$

such paths. Subtracting them from all $C(2n, n)$ paths gives the first formula. Finally,

$$C(2n, n-1) = \frac{n}{n+1}C(2n, n),$$

so the difference is $\frac{1}{n+1}C(2n, n)$. □

The first Catalan numbers are

$$C_0, C_1, C_2, C_3, C_4, C_5 = 1, 1, 2, 5, 14, 42.$$

For $n = 3$, the five Catalan step strings are

$$\text{RRRUUU}, \quad \text{RRURUU}, \quad \text{RRUURU}, \quad \text{RURRUU}, \quad \text{RURURU}.$$

4.7.1 Balanced-parenthesis strings

We replace every right step R by a left parenthesis and every up step U by a right parenthesis. A path ends at (n, n) exactly when the string contains n parentheses of each kind. The condition $y \leq x$ says that every initial segment contains at least as many left parentheses as right parentheses. This is exactly the condition for a balanced-parenthesis string.

Hence C_n also counts balanced strings of n pairs of parentheses. For example, the five paths above correspond to

$$((())), \quad (()()), \quad (())(), \quad ()(()), \quad ()()().$$

Conversely, replacing each left parenthesis by R and each right parenthesis by U recovers a unique Catalan path. The two replacements undo one another, so this correspondence is a bijection. This is an important combinatorial lesson: a named sequence is usually useful because it counts many families of objects, not only because it has a closed formula.

Remark 19 (A recurrence to revisit later). A first-return decomposition of a nonempty Catalan path gives

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}, \quad C_0 = 1.$$

A later note on combinatorial recurrences will develop this nonlinear recurrence. For now, the reflection argument supplies the closed formula.

Problem (Practice). How many northeast paths from $(0, 0)$ to $(4, 4)$ never go above $y = x$? How many go above the diagonal at least once? Which count comes from subtraction, and which comes from the reflection bijection?

4.8 Summary of the counting connections

The main formulas in these notes arise from a small number of reversible encodings and counting rules:

$$\begin{aligned}\text{size-}k \text{ multisets from } n \text{ types} &= C(n+k-1, k), \\ \text{solutions with lower bounds} &= C\left(N - \sum_i \ell_i + n - 1, n - 1\right), \\ (x+y)^n &= \sum_{k=0}^n C(n, k) x^k y^{n-k}, \\ C(n, k) &= C(n-1, k) + C(n-1, k-1), \\ \text{paths through } (c, d) &= C(c+d, c) C(a+b-c-d, a-c), \\ C_n &= \frac{1}{n+1} C(2n, n).\end{aligned}$$

In each case, the formula becomes easier to remember after identifying the objects behind it: shifted sequences, translated integer vectors, choices from labeled factors, subsets split by a distinguished element, concatenated paths, or reflected paths.

4.9 Further reading

The [course resources page](#) links to the books used as references for these notes. *Applied Combinatorics*, Sections 2.5–2.7, covers lattice paths and binomial and multinomial expansions. *Counting Rocks!*, Sections 4.1–4.3 and 4.5–4.6, covers Pascal’s recurrence, binomial identities, and multinomial counting. *Discrete Mathematics: An Open Introduction*, Sections 3.1–3.2, presents coefficient and lattice-path interpretations. *Combinatorics Through Guided Discovery*, Section 1.3.1, also discusses Catalan paths and the reflection principle.