Midterm 3 Practice Worksheet 4: Solutions

Combinatorics

Fall 2026

These are the solutions to Practice Worksheet 4. The write-ups model the amount of explanation expected on the exam.

Colorings of a flag

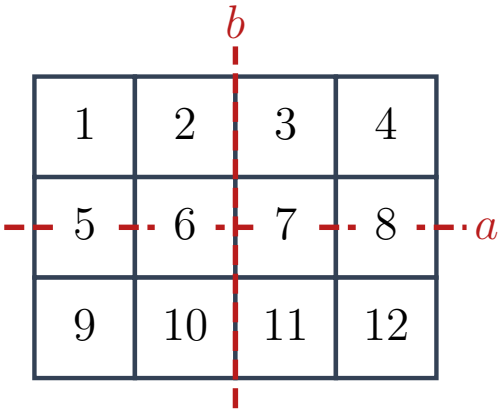


Figure 1: The cells of the flag, with the mirror lines of the reflections a and b.

Figure description: A rectangle of square cells in 3 rows and 4 columns, numbered 1 to 4 in the top row, 5 to 8 in the middle row, and 9 to 12 in the bottom row, from left to right. A dashed horizontal line labeled a runs through the middle row, and a dashed vertical line labeled b runs between the second and the third column.

(A) *For each symmetry, $g(x)$ is the cell where cell x ends up.*

- *The half turn sends each cell to the cell symmetric through the center:* $t = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(6\ 7)$. *No cell lies at the center.*
- *The reflection a exchanges the top and bottom rows and keeps each cell of the middle row in place:* $a = (1\ 9)(2\ 10)(3\ 11)(4\ 12)(5)(6)(7)(8)$.
- *The reflection b exchanges columns $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ in each row:* $b = (1\ 4)(2\ 3)(5\ 8)(6\ 7)(9\ 12)(10\ 11)$. *No cell lies on its mirror line.*

Element	Cycle type	$\mathrm{cyc}(g)$
e	$1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1$	12
t	$2, 2, 2, 2, 2, 2$	6
a	$2, 2, 2, 2, 1, 1, 1, 1$	8
b	$2, 2, 2, 2, 2, 2$	6

(B) A coloring is fixed by g exactly when it has one color on each cycle of g, so $|\mathrm{Fix}(g)| = 3^{\mathrm{cyc}(g)}$:

$$|\mathrm{Fix}(e)| = 3^{12} = 531441, \quad |\mathrm{Fix}(t)| = 3^6 = 729, \quad |\mathrm{Fix}(a)| = 3^8 = 6561, \quad |\mathrm{Fix}(b)| = 3^6 = 729.$$

For a: a coloring c is fixed by a when every cell has the same color as its mirror image, so cells 1 and 9 have the same color, and so do 2 and 10, 3 and 11, 4 and 12. The cells $5, 6, 7, 8$ of the middle row lie on the mirror line and can have any colors. That is 3^4 choices for the four pairs and 3^4 for the middle row: 3^8 colorings.

(C) By Burnside's lemma,

$$\frac{531441 + 729 + 6561 + 729}{4} = \frac{539460}{4} = 134865.$$

(D) With k colors,

$$\frac{k^{12} + k^6 + k^8 + k^6}{4} = \frac{k^{12} + k^8 + 2k^6}{4}.$$

For $k = 1$ this is $4/4 = 1$, and for $k = 3$ it gives 134865 again.

Colorings of a longer flag

1	2	3	4	5
6	7	8	9	10

Figure 2: The cells of the flag.

Figure description: A rectangle of square cells in 2 rows and 5 columns. The top row has cells 1 to 5 and the bottom row has cells 6 to 10, from left to right.

(A) For each symmetry, $g(x)$ is the cell where cell x ends up.

- $t = (1\ 10)(2\ 9)(3\ 8)(4\ 7)(5\ 6)$: each cell goes to the cell symmetric through the center, and no cell lies at the center.
- $a = (1\ 6)(2\ 7)(3\ 8)(4\ 9)(5\ 10)$: the two rows are exchanged.
- $b = (1\ 5)(2\ 4)(3)(6\ 10)(7\ 9)(8)$: the columns are exchanged in pairs, and the cells 3 and 8 of the middle column stay in place.

Element	Cycle type	$\mathrm{cyc}(g)$
e	$1,1,1,1,1,1,1,1,1,1$	10
t	$2,2,2,2,2$	5
a	$2,2,2,2,2$	5
b	$2,2,2,2,1,1$	6

(B) $|\mathrm{Fix}(g)| = 3^{\mathrm{cyc}(g)}$:

$$|\mathrm{Fix}(e)| = 3^{10} = 59049, \quad |\mathrm{Fix}(t)| = 3^5 = 243, \quad |\mathrm{Fix}(a)| = 3^5 = 243, \quad |\mathrm{Fix}(b)| = 3^6 = 729.$$

A coloring fixed by b gives each cell the same color as its mirror image: cells 1 and 5 have the same color, and so do 2 and 4, 6 and 10, 7 and 9. The cells 3 and 8 lie on the mirror line and can have any colors. That is 3^4 choices for the four pairs and 3^2 for the two middle cells, $3^6 = 729$ in all.

(C) By Burnside's lemma,

$$\frac{59049 + 243 + 243 + 729}{4} = \frac{60264}{4} = 15066.$$

Necklaces and bracelets

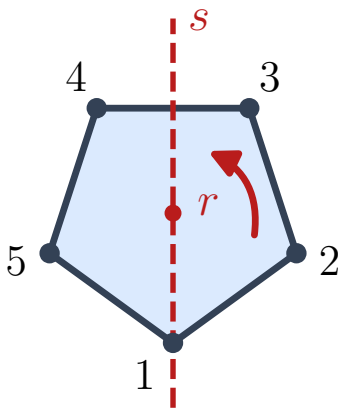


Figure 3: The labeled pentagon, the rotation r, and the mirror line of s.

Figure description: A regular pentagon with corner 1 at the bottom and the other corners counterclockwise: 2 at the lower right, 3 at the upper right, 4 at the upper left, and 5 at the lower left. A dashed vertical mirror line labeled s passes through corner 1 and the center, and a counterclockwise arrow labeled r turns about the center.

(A) The group is $C_5 = \{e, r, r^2, r^3, r^4\}$. The identity has 5 cycles and fixes $3^5 = 243$ colorings. Each of r, r^2, r^3, r^4 is a single 5-cycle (for example $r^2 = (1\ 3\ 5\ 2\ 4)$),

so $\mathrm{cyc} = 1$ and it fixes only the 3 one-color colorings.

$$\text{necklaces} = \frac{243 + 4 \cdot 3}{5} = \frac{255}{5} = 51.$$

(B) The group is D_5, with 10 elements. Each of the 5 reflections fixes the corner on its axis and exchanges the other four corners in two pairs; for example $s = (1)(2\ 5)(3\ 4)$. So $\mathrm{cyc} = 3$, and each reflection fixes $3^3 = 27$ colorings.

$$\text{bracelets} = \frac{243 + 4 \cdot 3 + 5 \cdot 27}{10} = \frac{390}{10} = 39.$$

(C) Every rotation is also an element of D_5, so two colorings in the same necklace are also in the same bracelet. Each bracelet is therefore a union of necklaces, and there are at most as many bracelets as necklaces.

Colorings of a square grid

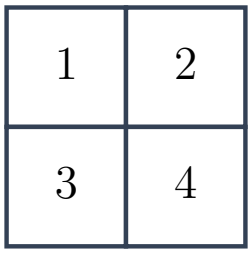


Figure 4: The cells of the grid.

Figure description: A 2 by 2 grid with cells 1 and 2 in the top row and cells 3 and 4 in the bottom row, from left to right.

(A) The quarter turn counterclockwise carries the top left cell to the bottom left, and so on: $r = (1\ 3\ 4\ 2)$. Then $r^2 = (1\ 4)(2\ 3)$ and $r^3 = (1\ 2\ 4\ 3)$.

Kind	How many	cyc	Fixed colorings	Product
identity	1	4	$3^4 = 81$	81
quarter turns r, r^3	2	1	3	6
half turn r^2	1	2	9	9
Sum	4	--	--	96

So there are $96/4 = 24$ colorings up to rotation.

(B) The reflections are $(1\ 2)(3\ 4)$ (vertical midline), $(1\ 3)(2\ 4)$ (horizontal midline), $(2\ 3)$ (diagonal through cells 1 and 4), and $(1\ 4)$ (diagonal through cells 2 and 3). The midline reflections have 2 cycles; the diagonal reflections fix the

two cells on the diagonal and have 3 cycles.

$$\frac{96 + 2 \cdot 3^2 + 2 \cdot 3^3}{8} = \frac{96 + 18 + 54}{8} = \frac{168}{8} = 21.$$

Colorings of solids

(A) The 12 rotations of the tetrahedron on the vertices: the identity (4 cycles), 8 vertex--face rotations (2 cycles each), and 3 edge--edge rotations (2 cycles each).

$$\frac{3^4 + 8 \cdot 3^2 + 3 \cdot 3^2}{12} = \frac{81 + 72 + 27}{12} = \frac{180}{12} = 15.$$

(B) The 24 rotations of the cube on the faces: identity (6 cycles); 6 face rotations by 90° or 270° (3 cycles); 3 face rotations by 180° (4 cycles); 8 vertex rotations (2 cycles); 6 edge rotations (3 cycles).

$$\frac{2^6 + 6 \cdot 2^3 + 3 \cdot 2^4 + 8 \cdot 2^2 + 6 \cdot 2^3}{24} = \frac{64 + 48 + 48 + 32 + 48}{24} = \frac{240}{24} = 10.$$

(C) On the vertices: identity (8 cycles); 90° face rotations (2 cycles); 180° face rotations (4 cycles); vertex rotations (4 cycles, counting the two fixed vertices on the axis); edge rotations (4 cycles).

$$\frac{2^8 + 6 \cdot 2^2 + 3 \cdot 2^4 + 8 \cdot 2^4 + 6 \cdot 2^4}{24} = \frac{256 + 24 + 48 + 128 + 96}{24} = \frac{552}{24} = 23.$$

(D) The 6 rotations of the prism on the faces: identity (5 cycles); 2 turns about the vertical axis (3 cycles: two fixed triangles and a 3-cycle of squares); 3 half turns (3 cycles: the triangles exchanged, one square fixed, two squares exchanged).

$$\frac{3^5 + 2 \cdot 3^3 + 3 \cdot 3^3}{6} = \frac{243 + 54 + 81}{6} = \frac{378}{6} = 63.$$

Strings up to reversal

(A) The group is $\{e, \rho\}$, where ρ reverses the string. On the positions $1, \dots, 5$, $\rho = (1\ 5)(2\ 4)(3)$ has 3 cycles; the fixed strings are the palindromes, $3^3 = 27$ of them. The identity fixes all $3^5 = 243$ strings. So there are $(243 + 27)/2 = 135$ strings up to reversal.
(B) On positions $1, \dots, 6$, $\rho = (1\ 6)(2\ 5)(3\ 4)$ has 3 cycles, so there are $(2^6 + 2^3)/2 =$

$(64 + 8)/2 = 36$ bit strings up to reversal.