Midterm 3 Practice Worksheet 3: Rotations of Solids

Combinatorics

Fall 2026

This worksheet is optional practice for Midterm 3. It is not collected or graded. Work on paper, as you would on the exam, and write the method that justifies each answer. Then compare with the solutions.

This worksheet covers the rotations of a solid: writing a rotation as a permutation of the vertices, sorting the rotations by their axes, and checking the count with the orbit–stabilizer theorem. The labels are different from the ones in the lecture notes and on the exam; use the labels in the figures. Permutations are composed from right to left, cycle types list the cycle lengths, including fixed points, in decreasing order, and $\mathrm{cyc}(g)$ is the number of cycles of g, including fixed points.

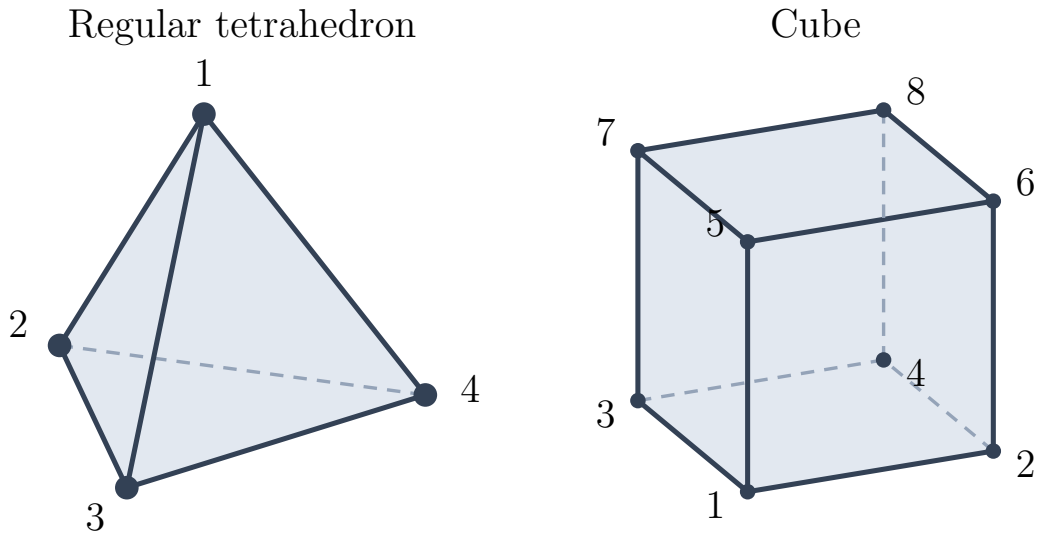


Figure 1: Left: a regular tetrahedron with top vertex 1 and base $2, 3, 4$, counterclockwise seen from above. Right: a cube whose bottom face has vertices 1 (front left), 2 (front right), 3 (back left), and 4 (back right); vertex $j + 4$ lies directly above vertex j. Dashed edges are hidden.

Figure description: Left: a regular tetrahedron with vertex 1 on top and base triangle 2, 3, 4; vertex 3 is at the front and the edge from 2 to 4 is hidden. Right: a cube whose bottom face has vertex 1 at the front left, 2 at the front right, 3 at the back left, and 4 at the back right; 5, 6, 7, 8 lie directly above 1, 2, 3, 4. Vertex 4 is hidden at the back.

1. (The rotations of the tetrahedron)

Besides the identity, every rotation of the regular tetrahedron is either a rotation by 120° or 240° about an axis through a vertex and the center of the opposite face (vertex–face), or a rotation by 180° about an axis through the midpoints of two opposite edges (edge–edge).

(A) Write in cycle notation, as permutations of the 4 vertices:

(i) the rotation by 180° about the axis through the midpoints of the edges 1–3 and 2–4;

(ii) the two rotations about the vertical axis through vertex 1;

(iii) the two rotations about the axis through vertex 2;

(iv) the rotation by 180° about the axis through the midpoints of the edges 1–4 and 2–3;

(v) the two rotations about the axis through vertex 3.

(B) Complete the table for the action on the 4 vertices.

Kind of rotation	**Axes**	**Rotations**	**Cycle type**	cyc
identity	–	1	$1, 1, 1, 1$	4
vertex–face, 120° or 240°				
edge–edge, 180°				
Total	–		–	–

(C) Find Stab(1) and the orbit of vertex 1, and check that your total equals $|\text{orbit of vertex } 1| \cdot |\text{Stab}(1)|$. Then do the same for vertex 3.

(D) Complete the same table for the action on the 6 edges. (Write the edge with ends a and b as ab.)

2. (The rotations of the cube)

Besides the identity, every rotation of the cube is a rotation by 90°, 180°, or 270° about an axis through the centers of two opposite faces (face–face), a rotation by 120° or 240° about an axis through two opposite vertices (vertex–vertex), or a rotation by 180° about an axis through the midpoints of two opposite edges (edge–edge).

(A) Write in cycle notation, as permutations of the 8 vertices:

(i) the rotation by 90° about the vertical axis that sends vertex 1 to vertex 2;

(ii) the rotation by 180° about the axis through the midpoints of the edges 1–2 and 7–8;

(iii) the rotation by 120° about the axis through the vertices 1 and 8 that sends vertex 2 to vertex 3;

(iv) the rotation by 180° about the axis through the centers of the front face 1 2 6 5 and the back face 3 4 8 7;

(v) the rotation by 90° about the axis through the centers of the left face 1 3 7 5 and the right face 2 4 8 6 that sends vertex 1 to vertex 5;

(vi) the rotation by 180° about the axis through the midpoints of the edges 1–5 and 4–8.

(B) Complete the table for the action on the 8 vertices, and check that the total number of rotations is 24.

Kind of rotation	**Axes**	**Rotations**	**Cycle type**	cyc
identity	–	1	$1, 1, 1, 1, 1, 1, 1, 1$	8
face–face, 90° or 270°				
face–face, 180°				
vertex–vertex, 120° or 240°				
edge–edge, 180°				
Total	–		–	–

(C) Check the total with the orbit–stabilizer theorem twice: once with the orbit and stabilizer of vertex 1, and once with the orbit and stabilizer of the bottom face.

(D) Complete the same table for the action on the 6 faces.

Hint. To count the rotations of one kind, count the axes of that kind and multiply by the number of nonidentity rotations about each axis. Rotations of the same kind have the same cycle type, so you only need the cycles of one rotation of each kind. In this cube, opposite vertices add up to 9, and a rotation sends opposite vertices to opposite vertices.

3. (The rotations of a triangular prism)

A triangular prism has two equilateral triangles as ends and three squares as sides. Its top triangle has vertices $1, 2, 3$, counterclockwise seen from above, and vertex $j + 3$ lies directly below vertex j.

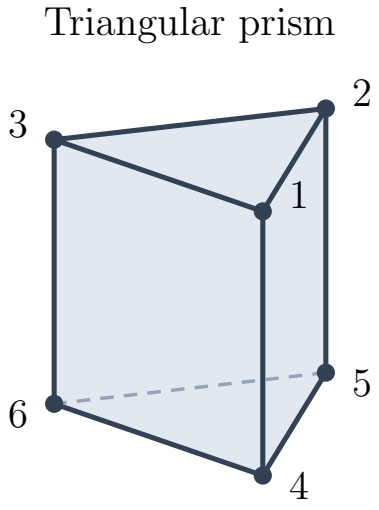


Figure 2: A triangular prism with top triangle $1, 2, 3$ and bottom triangle $4, 5, 6$; vertex $j + 3$ lies below vertex j. Dashed edges are hidden.

Figure description: A triangular prism standing on a triangle. The top triangle has vertices 1 at the front, 2 at the back right, and 3 at the back left; 4, 5, 6 lie directly below 1, 2, 3. The edge from 5 to 6 is hidden.

(A) Besides the identity, the rotations of the prism are the rotations by 120° and 240° about the vertical axis through the centers of the two triangles, and the half turns about a horizontal axis through the midpoint of a vertical edge and the center of the opposite square. Write all the rotations in cycle notation, as permutations of the 6 vertices. How many are there?

(B) Make a table, as in the previous problems, for the action on the 6 vertices and for the action on the 5 faces.

(C) Check the number of rotations with the orbit–stabilizer theorem, using the orbit and stabilizer of the top triangle.