Midterm 3 Practice Worksheet 3: Solutions

Combinatorics

Fall 2026

These are the solutions to Practice Worksheet 3. The write-ups model the amount of explanation expected on the exam. The vertex labels are those of the figures on the worksheet: the tetrahedron has top vertex 1 and base $2, 3, 4$; the cube has bottom face 1 (front left), 2 (front right), 3 (back left), 4 (back right), with $j + 4$ above j; the prism has top triangle $1, 2, 3$, with $j + 3$ below j.

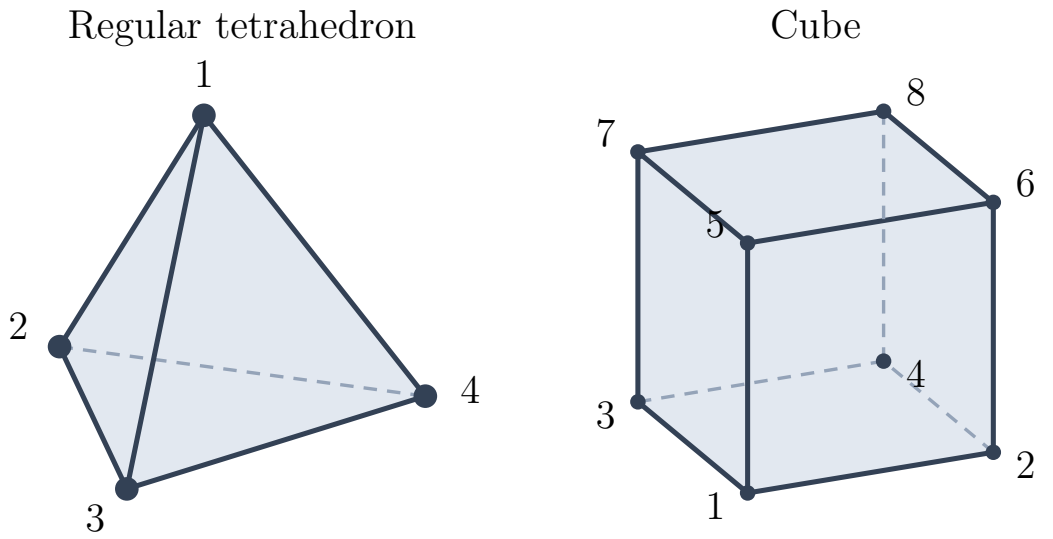


Figure 1: Left: a regular tetrahedron with top vertex 1 and base $2, 3, 4$, counterclockwise seen from above. Right: a cube whose bottom face has vertices 1 (front left), 2 (front right), 3 (back left), and 4 (back right); vertex $j + 4$ lies directly above vertex j. Dashed edges are hidden.

Figure description: Left: a regular tetrahedron with vertex 1 on top and base triangle 2, 3, 4; vertex 3 is at the front and the edge from 2 to 4 is hidden. Right: a cube whose bottom face has vertex 1 at the front left, 2 at the front right, 3 at the back left, and 4 at the back right; 5, 6, 7, 8 lie directly above 1, 2, 3, 4. Vertex 4 is hidden at the back.

The rotations of the tetrahedron

(A) (i) The half turn about the axis through the midpoints of $1{-}{-}3$ and $2{-}{-}4$ exchanges the two ends of each of these edges:

$$(1\ 3)(2\ 4).$$

(ii) The vertical axis passes through vertex 1 and the center of the base. Turning by 120° counterclockwise (seen from above) moves the base vertices $2 \mapsto 3 \mapsto 4 \mapsto 2$ and fixes 1; turning by 240° moves them the other way. The two rotations are $(2\ 3\ 4)$ and $(2\ 4\ 3)$.
(iii) The axis through vertex 2 passes through the center of the opposite face $1, 3, 4$, and the rotations turn that face: $(1\ 3\ 4)$ and $(1\ 4\ 3)$.

(iv) The half turn exchanges the ends of the edges on its axis: $(1\ 4)(2\ 3)$.
(v) The axis through vertex 3 passes through the center of the opposite face $1, 2, 4$, and the rotations about it are $(1\ 2\ 4)$ and $(1\ 4\ 2)$.
(B) There is one vertex--face axis through each vertex, so 4 axes, each with 2 nonidentity rotations: 8 rotations, each fixing the vertex on its axis and moving the other three in a 3-cycle. The 6 edges form 3 pairs of opposite edges (12 and 34, 13 and 24, 14 and 23), so there are 3 edge--edge axes, each with one half turn: 3 rotations, each a product of two 2-cycles, as in (A)(i).

Kind of rotation	Axes	Rotations	Cycle type	cyc
identity	--	1	$1, 1, 1, 1$	4
vertex--face, $120°$ or $240°$	4	8	$3, 1$	2
edge--edge, $180°$	3	3	$2, 2$	2
Total	--	12	--	--

(C) The rotations fixing vertex 1 are those about the vertical axis: $\mathrm{Stab}(1) = \{e, (2\ 3\ 4), (2\ 4\ 3)\}$. Vertex 1 can be sent to every vertex; for example, the half turns $(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, $(1\ 4)(2\ 3)$ send it to 2, 3, 4. So the orbit of vertex 1 is $\{1, 2, 3, 4\}$, and $4 \cdot 3 = 12$, which agrees with the table.
For vertex 3: the rotations fixing 3 are those about the axis through 3, so $\mathrm{Stab}(3) = \{e, (1\ 2\ 4), (1\ 4\ 2)\}$ by (A)(v). The half turns $(1\ 3)(2\ 4)$, $(1\ 4)(2\ 3)$, $(1\ 2)(3\ 4)$ send 3 to 1, 2, 4, so the orbit of vertex 3 is $\{1, 2, 3, 4\}$, and again $4 \cdot 3 = 12$.
(D) Take one rotation of each kind. The rotation $(2\ 3\ 4)$ moves the edges at the top $12 \mapsto 13 \mapsto 14 \mapsto 12$ and the base edges $23 \mapsto 34 \mapsto 24 \mapsto 23$: two 3-cycles. The half turn $(1\ 3)(2\ 4)$ sends the edges 13 and 24 to themselves (each is flipped end to end, but as an edge it stays in place) and exchanges $12 \leftrightarrow 34$ and $14 \leftrightarrow 23$: two fixed edges and two 2-cycles.

Kind of rotation	Axes	Rotations	Cycle type	cyc
identity	--	1	$1, 1, 1, 1, 1, 1$	6
vertex--face, $120°$ or $240°$	4	8	$3, 3$	2
edge--edge, $180°$	3	3	$2, 2, 1, 1$	4
Total	--	12	--	--

The rotations of the cube

(A) (i) Going around the bottom face, the vertices are 1 (front left), 2 (front right), 4 (back right), 3 (back left). The quarter turn that sends 1 to 2 moves each bottom vertex one step along this cycle, and each top vertex follows the vertex below it:

$$(1\ 2\ 4\ 3)(5\ 6\ 8\ 7).$$

(ii) The edge opposite 1--2 is 7--8 (opposite vertices add up to 9). The half turn exchanges the two ends of each of these edges, $1 \leftrightarrow 2$ and $7 \leftrightarrow 8$. It also exchanges the two faces that meet along the edge 1--2: the bottom face 1243 and the front face 1265. Vertex 3, next to 1 in the bottom face, goes to the vertex next to 2 in the front face, which is 6; vertex 4, next to 2 in the bottom face, goes to 5, next to 1 in the front face. A half turn has order 2, so

$$(1\ 2)(3\ 6)(4\ 5)(7\ 8).$$

(iii) The vertices 1 and 8 on the axis stay fixed. The three neighbors $2, 3, 5$ of vertex 1 are turned among themselves; since $2 \mapsto 3$, they go $2 \mapsto 3 \mapsto 5 \mapsto 2$. Opposite vertices go to opposite vertices: the vertices opposite $2, 3, 5$ are $7, 6, 4$, so $7 \mapsto 6 \mapsto 4 \mapsto 7$. So

$$(1)(8)(2\ 3\ 5)(4\ 7\ 6).$$

(iv) The half turn keeps the front face and the back face in place and turns each by 180°, which exchanges opposite corners of each face:

$$(1\ 6)(2\ 5)(3\ 8)(4\ 7).$$

(v) Going around the left face, the vertices are 1 (bottom front), 5 (top front), 7 (top back), 3 (bottom back). The quarter turn that sends 1 to 5 moves each of them one step along this cycle, and each vertex of the right face follows its neighbor on the left face (2 is next to 1, 6 to 5, 8 to 7, 4 to 3):

$$(1\ 5\ 7\ 3)(2\ 6\ 8\ 4).$$

(vi) The edges 1--5 and 4--8 are opposite (opposite vertices add up to 9). The half turn exchanges the ends of each, $1 \leftrightarrow 5$ and $4 \leftrightarrow 8$, and exchanges the two faces that meet along 1--5: the front face 1265 and the left face 1375. Vertex 2, next to 1 in the front face, goes to the vertex next to 5 in the left face, which is 7; vertex 6, next to 5 in the front face, goes to 3, next to 1 in the left face. So

$$(1\ 5)(2\ 7)(3\ 6)(4\ 8).$$

(B) There are 3 face--face axes (the 6 faces form 3 opposite pairs), each with rotations by $90°$, $180°$, $270°$; 4 vertex--vertex axes (the 8 vertices form 4 opposite pairs), each with 2 nonidentity rotations; and 6 edge--edge axes (the 12 edges form 6 opposite pairs), each with one half turn. The cycle types come from the examples in (A): (i) for $90°$, (iv) for $180°$ about a face axis, (iii) for the vertex axes, and (ii) for the edge axes.

Kind of rotation	Axes	Rotations	Cycle type	cyc
identity	--	1	$1,1,1,1,1,1,1,1$	8
face--face, $90°$ or $270°$	3	6	$4,4$	2
face--face, $180°$	3	3	$2,2,2,2$	4
vertex--vertex, $120°$ or $240°$	4	8	$3,3,1,1$	4
edge--edge, $180°$	6	6	$2,2,2,2$	4
Total	--	24	--	--

Indeed $1+6+3+8+6=24$.

(C) A rotation fixing vertex 1 also fixes the opposite vertex 8, so it is a rotation about the axis through 1 and 8: $\mathrm{Stab}(1)$ has 3 elements ($0°$, $120°$, $240°$). Vertex 1 can be carried to any of the 8 vertices, and $8 \cdot 3 = 24$.

A rotation keeping the bottom face in place is a rotation about the vertical axis: 4 of them. The bottom face can be carried to any of the 6 faces, and $6 \cdot 4 = 24$.

(D) Use the same four examples.

- (i), $90°$ about the vertical axis: the bottom and top faces stay, and the side faces form the 4-cycle (front right back left).
- (iv), $180°$ about the front--back axis: the front and back faces stay, the bottom and top are exchanged, and the left and right are exchanged.
- (iii), $120°$ about the axis through 1 and 8: the three faces at vertex 1 (bottom, left, front) form a 3-cycle, and the three faces at vertex 8 (top, right, back) form another.
- (ii), $180°$ about an edge axis: the faces are exchanged in three pairs, bottom with front, top with back, and left with right; no face is fixed.

Kind of rotation	Axes	Rotations	Cycle type	cyc
identity	--	1	$1,1,1,1,1,1$	6
face--face, 90° or 270°	3	6	$4,1,1$	3
face--face, 180°	3	3	$2,2,1,1$	4
vertex--vertex, 120° or 240°	4	8	$3,3$	2
edge--edge, 180°	6	6	$2,2,2$	3
Total	--	24	--	--

Notice that the face and edge half turns have the same cycle type on the vertices but different cycle types on the faces.

The rotations of a triangular prism

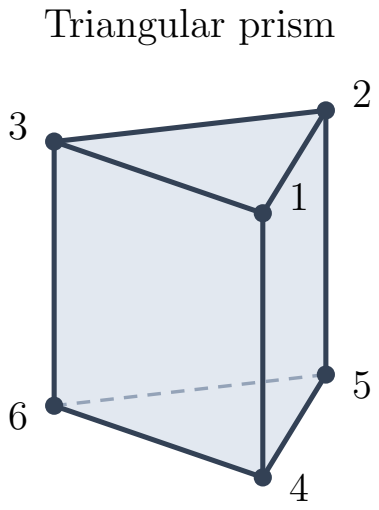


Figure 2: A triangular prism with top triangle $1,2,3$ and bottom triangle $4,5,6$; vertex $j+3$ lies below vertex j. Dashed edges are hidden.

Figure description: A triangular prism standing on a triangle. The top triangle has vertices 1 at the front, 2 at the back right, and 3 at the back left; 4, 5, 6 lie directly below 1, 2, 3. The edge from 5 to 6 is hidden.

(A) About the vertical axis: $(1\,2\,3)(4\,5\,6)$ and $(1\,3\,2)(4\,6\,5)$. For the half turns, take the axis through the midpoint of the vertical edge 1--4 and the center of the opposite square $2\,3\,6\,5$. The half turn flips the edge 1--4 end to end, so $1 \leftrightarrow 4$, and turns the square $2\,3\,6\,5$ by 180° about its center, which exchanges its opposite corners, $2 \leftrightarrow 6$ and $3 \leftrightarrow 5$. In the same way for the other two vertical edges:

$$(1\,4)(2\,6)(3\,5), \qquad (2\,5)(1\,6)(3\,4), \qquad (3\,6)(1\,5)(2\,4).$$

With the identity, there are $1 + 2 + 3 = 6$ rotations.
(B) On the vertices, the turns about the vertical axis have two 3-cycles, and the half turns have three 2-cycles. On the faces, a turn about the vertical axis keeps both triangles and moves the three squares in a 3-cycle; a half turn exchanges the two triangles, keeps the square on its axis in place, and exchanges the other two squares.

Kind of rotation	Axes	Rotations	Type on vertices	cyc	Type on faces	cyc
identity	--	1	$1, 1, 1, 1, 1, 1$	6	$1, 1, 1, 1, 1$	5
vertical, 120° or 240°	1	2	$3, 3$	2	$3, 1, 1$	3
half turn	3	3	$2, 2, 2$	3	$2, 2, 1$	3
Total	--	6	--	--	--	--

(C) The top triangle can be kept in place or exchanged with the bottom triangle, so its orbit has 2 elements. The rotations keeping it in place are the 3 rotations about the vertical axis. So the prism has $2 \cdot 3 = 6$ rotations, as in (A).