Midterm 3 Practice Worksheet 2: Orbits, Stabilizers, and Fixed Sets

Combinatorics

Fall 2026

This worksheet is optional practice for Midterm 3. It is not collected or graded. Work on paper, as you would on the exam, and write the method that justifies each answer. Then compare with the solutions.

This worksheet covers group actions: the true-or-false facts about orbits and stabilizers, and the computation of orbits, stabilizers, and fixed sets of colorings. As on the exam, a symmetry g moves a coloring c by carrying the color at each corner x to the corner $g(x)$; the result is written $g \circlearrowright c$ (the lecture notes write $g \cdot c$). The orbit of c under a group G is $\mathrm{Orb}_G(c)$.

The square. Label the corners of a square $1, 2, 3, 4$ counterclockwise, starting at the bottom left. Its eight symmetries form the group D_4:

$$\begin{array}{llll} e, & r = (1\ 2\ 3\ 4), & r^2 = (1\ 3)(2\ 4), & r^3 = (1\ 4\ 3\ 2), \\ d_{13} = (2\ 4), & v = (1\ 2)(3\ 4), & d_{24} = (1\ 3), & h = (1\ 4)(2\ 3). \end{array}$$

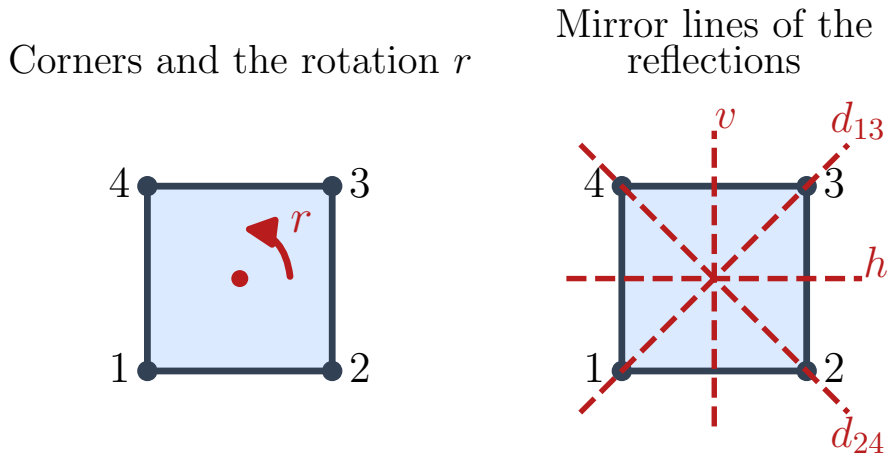


Figure 1: The labeled square. Left: the corners and the rotation r by 90° counterclockwise. Right: the mirror lines of the reflections v, h, d_{13}, and d_{24}.

Figure description: Two copies of a square with corner 1 at the bottom left, 2 at the bottom right, 3 at the top right, and 4 at the top left. The left copy has a counterclockwise arrow labeled r about the center. The right copy has four dashed mirror lines: v vertical, h horizontal, d13 through corners 1 and 3, and d24 through corners 2 and 4.

A coloring is written as the string $c(1)c(2)c(3)c(4)$. For example, $r \circlearrowright$`BBBW` $=$ `WBBB`.

1. (True or false)

Decide whether each statement is always true, and give a one-line reason or a counterexample. In each statement, a finite group G acts on a finite set Y.

(A) For every $y \in Y$, the stabilizer $\mathrm{Stab}(y)$ contains e, and if $g, h \in \mathrm{Stab}(y)$, then $gh \in \mathrm{Stab}(y)$.

(B) For every $g \in G$, the fixed set $\mathrm{Fix}(g)$ is a subgroup of G.

(C) If $|G| = 10$, an orbit can have 4 elements.

(D) If $|G| = 10$, an orbit can have 5 elements.

(E) If $z \in \mathrm{Orb}_G(y)$, then $\mathrm{Orb}_G(z) = \mathrm{Orb}_G(y)$.

(F) Two different orbits can have exactly one element in common.

(G) If $|G| = 8$ and $|Y| = 5$, then some element of Y is fixed by every element of G.

(H) If $\mathrm{Stab}(y) = \{e\}$, then $\mathrm{Orb}_G(y)$ has $|G|$ elements.

(I) When D_4 acts on the corners of the square, all four corners have the same stabilizer.

(J) $|\mathrm{Fix}(e)| = |Y|$.

2. (Colorings of the square)

(A) Let $c =$ `BBBW`. Compute $g \circlearrowleft c$ for each of the eight symmetries $g \in D_4$. Then list the orbit $\mathrm{Orb}_{D_4}(c)$ and the stabilizer $\mathrm{Stab}(c)$, and check that $|\mathrm{Orb}_{D_4}(c)| \cdot |\mathrm{Stab}(c)| = |D_4|$.

(B) Now use three colors, black (B), white (W), and red (R). Let $c =$ `RBBW`. Compute $g \circlearrowleft c$ for all eight symmetries, and find $\mathrm{Stab}(c)$ and $\mathrm{Orb}_{D_4}(c)$. Explain without computing why the stabilizer must be so small.

(C) For black-and-white colorings, list the colorings in $\mathrm{Fix}(v)$, in $\mathrm{Fix}(r^3)$, and in $\mathrm{Fix}(d_{13})$. Explain how each number of colorings follows from the cycles of v, r^3, and d_{13}.

(D) Still with three colors, let $c =$ `RBRW`. Compute $g \circlearrowleft c$ for each of the eight symmetries g. Then list $\mathrm{Orb}_{D_4}(c)$ and $\mathrm{Stab}(c)$, and check that $|\mathrm{Orb}_{D_4}(c)| \cdot |\mathrm{Stab}(c)| = |D_4|$.

(E) For black-and-white colorings, list the colorings in $\mathrm{Fix}(d_{24})$ and in $\mathrm{Fix}(r^2)$. Explain how the numbers of colorings follow from the cycles of d_{24} and of r^2.

3. (Colorings of the hexagon)

Label the corners of a regular hexagon $1, \ldots, 6$ counterclockwise, starting at the bottom, as in the figure. Its twelve symmetries form the group D_6:

e,	$r = (1\ 2\ 3\ 4\ 5\ 6)$,	$r^2 = (1\ 3\ 5)(2\ 4\ 6)$,
$r^3 = (1\ 4)(2\ 5)(3\ 6)$,	$r^4 = (1\ 5\ 3)(2\ 6\ 4)$,	$r^5 = (1\ 6\ 5\ 4\ 3\ 2)$,
$s = (2\ 6)(3\ 5)$,	$rs = (1\ 2)(3\ 6)(4\ 5)$,	$r^2 s = (1\ 3)(4\ 6)$,
$r^3 s = (1\ 4)(2\ 3)(5\ 6)$,	$r^4 s = (1\ 5)(2\ 4)$,	$r^5 s = (1\ 6)(2\ 5)(3\ 4)$.

Black-and-white colorings are written as strings $c(1)c(2)\cdots c(6)$.

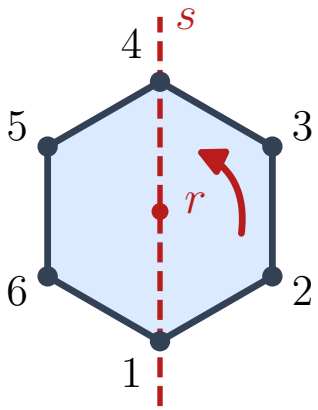


Figure 2: The labeled hexagon, the rotation r, and the mirror line of s.

Figure description: A regular hexagon with corner 1 at the bottom and the other corners counterclockwise: 2 at the lower right, 3 at the upper right, 4 at the top, 5 at the upper left, and 6 at the lower left. A dashed vertical mirror line labeled s passes through corners 1 and 4, and a counterclockwise arrow labeled r turns about the center.

(A) Let $c' =$ `BBWBBW`. Find $\mathrm{Stab}(c')$.

(B) Use the orbit–stabilizer theorem to find the number of colorings in $\mathrm{Orb}_{D_6}(c')$, and then list them.

(C) For each of $g = r^2$, r^3, s, and rs, find $|\mathrm{Fix}(g)|$ for the action on the $2^6 = 64$ black-and-white colorings, and explain your count using the cycles of g.

(D) Let $c'' =$ `BWBWBW`. Find $\mathrm{Stab}(c'')$. Use the orbit–stabilizer theorem to find the number of colorings in $\mathrm{Orb}_{D_6}(c'')$, and then list them.

Hint. A symmetry fixes a black-and-white coloring exactly when it sends the set of black corners to itself. A coloring is fixed by g exactly when it has the same color on all the corners of each cycle of g; fixed corners are cycles of length 1 and can have any color.

4. (Orbit–stabilizer in other settings)

(A) The rotations of a regular tetrahedron act on its 4 faces, and every face can be carried to every other face. Find the number of rotations that carry a given face onto itself, and use the orbit–stabilizer theorem to find the number of rotations of the tetrahedron.

(B) The 24 rotations of a cube act on its 12 edges, and every edge can be carried to every other edge. Use the orbit–stabilizer theorem to find the number of rotations that carry a given edge onto itself, and describe them.

(C) A triangular prism has two equilateral triangles as ends and three squares as sides. Its rotations act on its 6 vertices, and every vertex can be carried to every other vertex. Show that only the identity fixes a vertex, and find the number of rotations.

(D) D_4 acts on the six 2-element subsets of the corners of the square by $g \circlearrowright \{a, b\} = \{g(a), g(b)\}$. Find the orbits of this action. Find $\mathrm{Stab}(\{1, 2\})$ and $\mathrm{Stab}(\{1, 3\})$, and check the orbit–stabilizer theorem for both.