Midterm 3 Practice Worksheet 2: Solutions

Combinatorics

Fall 2026

These are the solutions to Practice Worksheet 2. The write-ups model the amount of explanation expected on the exam. A symmetry g moves a coloring c by carrying the color at each corner x to the corner $g(x)$; the result is $g \circlearrowleft c$.

True or false

(A) True. $e \circlearrowleft y = y$, so $e \in \mathrm{Stab}(y)$. If $g \circlearrowleft y = y$ and $h \circlearrowleft y = y$, then $(gh) \circlearrowleft y = g \circlearrowleft (h \circlearrowleft y) = g \circlearrowleft y = y$. (These two facts make $\mathrm{Stab}(y)$ a subgroup of G.)
(B) False. $\mathrm{Fix}(g)$ is a set of elements of Y, not of G. For example, when D_4 acts on colorings of the square, $\mathrm{Fix}(r^2) = \{\texttt{BBBB}, \texttt{BWBW}, \texttt{WBWB}, \texttt{WWWW}\}$ is a set of colorings.
(C) False. By the orbit--stabilizer theorem, $|\mathrm{Orb}_G(y)| \cdot |\mathrm{Stab}(y)| = 10$, so every orbit size divides 10, and 4 does not.
(D) True. For example, D_5 has 10 elements and acts on the 5 corners of the regular pentagon with a single orbit of 5 corners.
(E) True. Write $z = g \circlearrowleft y$. Then $k \circlearrowleft z = (kg) \circlearrowleft y$ and $k \circlearrowleft y = (kg^{-1}) \circlearrowleft z$ for every $k \in G$, so each orbit is contained in the other.
(F) False. By (E), two orbits with a common element z are both equal to $\mathrm{Orb}_G(z)$. Different orbits are disjoint.
(G) True. The orbit sizes divide 8, so they are 1, 2, 4, or 8, and they add up to 5. Since 5 is odd, some orbit has odd size, and the only odd choice is 1. An orbit $\{y\}$ means $g \circlearrowleft y = y$ for every $g \in G$.
(H) True. $|\mathrm{Orb}_G(y)| = |G|/|\mathrm{Stab}(y)| = |G|/1 = |G|$.
(I) False. $\mathrm{Stab}(1) = \{e, d_{13}\}$, while $\mathrm{Stab}(2) = \{e, d_{24}\}$. The stabilizers of corners in one orbit have the same size, but they need not be equal.
(J) True. $e \circlearrowleft y = y$ for every $y \in Y$.

Colorings of the square

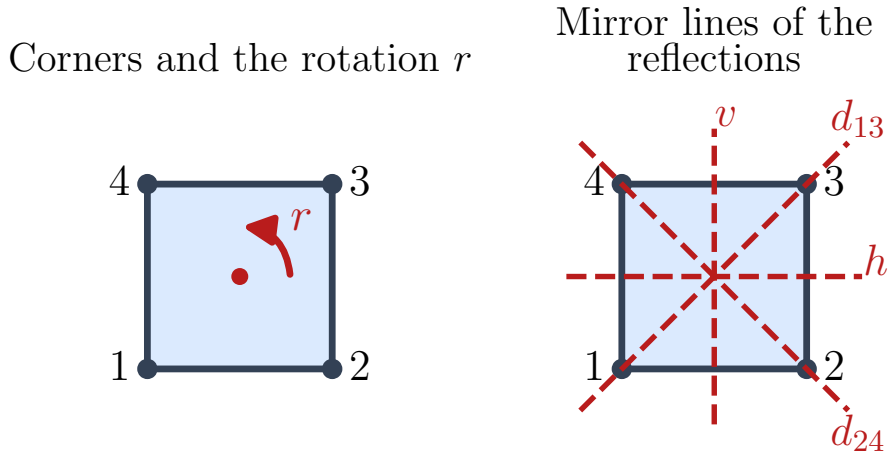


Figure 1: The labeled square. Left: the corners and the rotation r by 90° counterclockwise. Right: the mirror lines of the reflections v, h, d_{13}, and d_{24}.

Figure description: Two copies of a square with corner 1 at the bottom left, 2 at the bottom right, 3 at the top right, and 4 at the top left. The left copy has a counterclockwise arrow labeled r about the center. The right copy has four dashed mirror lines: v vertical, h horizontal, d13 through corners 1 and 3, and d24 through corners 2 and 4.

(A) For each g, the color at corner x moves to corner $g(x)$. For example, $v = (1\ 2)(3\ 4)$ moves the colors of corners $1, 2, 3, 4$ (B, B, B, W) to corners $2, 1, 4, 3$, which gives BBWB. All eight results:

$$e\circlearrowleft c = \texttt{BBBW}, \quad r\circlearrowleft c = \texttt{WBBB}, \quad r^2\circlearrowleft c = \texttt{BWBB}, \quad r^3\circlearrowleft c = \texttt{BBWB},$$
$$d_{13}\circlearrowleft c = \texttt{BWBB}, \quad v\circlearrowleft c = \texttt{BBWB}, \quad d_{24}\circlearrowleft c = \texttt{BBBW}, \quad h\circlearrowleft c = \texttt{WBBB}.$$

The different results form the orbit

$$\mathrm{Orb}_{D_4}(c) = \{\texttt{BBBW}, \texttt{WBBB}, \texttt{BWBB}, \texttt{BBWB}\},$$

the four colorings with exactly one white corner. The results equal to c occur for $g = e$ and $g = d_{24} = (1\ 3)$, so $\mathrm{Stab}(c) = \{e, d_{24}\}$: these are the symmetries that keep the white corner 4 in place. Check: $|\mathrm{Orb}_{D_4}(c)| \cdot |\mathrm{Stab}(c)| = 4 \cdot 2 = 8 = |D_4|$.

(B) In the same way:

$$e\circlearrowleft c = \texttt{RBBW}, \quad r\circlearrowleft c = \texttt{WRBB}, \quad r^2\circlearrowleft c = \texttt{BWRB}, \quad r^3\circlearrowleft c = \texttt{BBWR},$$
$$d_{13}\circlearrowleft c = \texttt{RWBB}, \quad v\circlearrowleft c = \texttt{BRWB}, \quad d_{24}\circlearrowleft c = \texttt{BBRW}, \quad h\circlearrowleft c = \texttt{WBBR}.$$

All eight colorings are different, so $\mathrm{Orb}_{D_4}(c)$ consists of these 8 colorings and $\mathrm{Stab}(c) = \{e\}$.

Without computing: a symmetry fixing c must keep the only red corner 1 and the only white corner 4 in place. These corners are adjacent, and a symmetry of the square is determined by where it sends corner 1 and one neighbor of 1. So only e fixes c, and the orbit has $8/1 = 8$ colorings.

(C) A coloring is fixed by g exactly when it has one color on each cycle of g, so $|\mathrm{Fix}(g)| = 2^{\mathrm{cyc}(g)}$.

- $v = (1\ 2)(3\ 4)$ has 2 cycles: corners $1, 2$ agree and corners $3, 4$ agree. $\mathrm{Fix}(v) = \{\mathtt{BBBB}, \mathtt{BBWW}, \mathtt{WWBB}, \mathtt{WWWW}\}$, $2^2 = 4$ colorings.
- $r^3 = (1\ 4\ 3\ 2)$ has 1 cycle: all corners agree. $\mathrm{Fix}(r^3) = \{\mathtt{BBBB}, \mathtt{WWWW}\}$, $2^1 = 2$ colorings.
- $d_{13} = (1)(3)(2\ 4)$ has 3 cycles: corners $2, 4$ agree, and corners 1 and 3 are free.

  $$\mathrm{Fix}(d_{13}) = \{\mathtt{BBBB}, \mathtt{BBWB}, \mathtt{BWBW}, \mathtt{BWWW}, \mathtt{WBBB}, \mathtt{WBWB}, \mathtt{WWBW}, \mathtt{WWWW}\},$$

  $2^3 = 8$ colorings. The fixed corners 1 and 3 are cycles too; forgetting them would give the wrong count $2^1 = 2$.

(D) The color at corner x moves to corner $g(x)$. For example, r moves the colors R, B, R, W of corners $1, 2, 3, 4$ to corners $2, 3, 4, 1$, which gives $\mathtt{WRBR}$. All eight results:

$$\begin{aligned} e\circlearrowleft c &= \mathtt{RBRW}, & r\circlearrowleft c &= \mathtt{WRBR}, & r^2\circlearrowleft c &= \mathtt{RWRB}, & r^3\circlearrowleft c &= \mathtt{BRWR}, \\ d_{13}\circlearrowleft c &= \mathtt{RWRB}, & v\circlearrowleft c &= \mathtt{BRWR}, & d_{24}\circlearrowleft c &= \mathtt{RBRW}, & h\circlearrowleft c &= \mathtt{WRBR}. \end{aligned}$$

So $\mathrm{Orb}_{D_4}(c) = \{\mathtt{RBRW}, \mathtt{WRBR}, \mathtt{RWRB}, \mathtt{BRWR}\}$ and $\mathrm{Stab}(c) = \{e, d_{24}\}$: the reflection $d_{24} = (1\ 3)$ exchanges the two red corners and fixes the black corner 2 and the white corner 4. Check: $|\mathrm{Orb}_{D_4}(c)| \cdot |\mathrm{Stab}(c)| = 4 \cdot 2 = 8 = |D_4|$.

(E) As in (C), $|\mathrm{Fix}(g)| = 2^{\mathrm{cyc}(g)}$. $d_{24} = (1\ 3)(2)(4)$ has 3 cycles: corners 1 and 3 agree, and corners 2 and 4 are free. There are $2^3 = 8$ fixed colorings:

$$\mathrm{Fix}(d_{24}) = \{\mathtt{BBBB}, \mathtt{BBBW}, \mathtt{BWBB}, \mathtt{BWBW}, \mathtt{WBWB}, \mathtt{WBWW}, \mathtt{WWWB}, \mathtt{WWWW}\}.$$

$r^2 = (1\ 3)(2\ 4)$ has 2 cycles: corners $1, 3$ agree and corners $2, 4$ agree. There are $2^2 = 4$ fixed colorings:

$$\mathrm{Fix}(r^2) = \{\mathtt{BBBB}, \mathtt{BWBW}, \mathtt{WBWB}, \mathtt{WWWW}\}.$$

Colorings of the hexagon

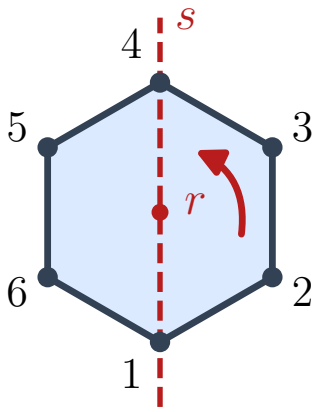


Figure 2: The labeled hexagon, the rotation r, and the mirror line of s.

Figure description: A regular hexagon with corner 1 at the bottom and the other corners counterclockwise: 2 at the lower right, 3 at the upper right, 4 at the top, 5 at the upper left, and 6 at the lower left. A dashed vertical mirror line labeled s passes through corners 1 and 4, and a counterclockwise arrow labeled r turns about the center.

(A) A symmetry fixes $c' =$ `BBWBBW` exactly when it sends the set of white corners $\{3, 6\}$ to itself. Check each element:

- rotations: e fixes everything; r^3 exchanges $3 \leftrightarrow 6$; r, r^2, r^4, r^5 send 3 to 4, 5, 1, 2, none of which is white.
- reflections: $rs = (1\ 2)(3\ 6)(4\ 5)$ exchanges $3 \leftrightarrow 6$, and $r^4 s = (1\ 5)(2\ 4)$ fixes 3 and 6; the others send 3 elsewhere (s to 5, $r^2 s$ to 1, $r^3 s$ to 2, $r^5 s$ to 4).

So $\mathrm{Stab}(c') = \{e, r^3, rs, r^4 s\}$.
(B) By the orbit--stabilizer theorem, $|\mathrm{Orb}_{D_6}(c')| = 12/4 = 3$. The white corners $3, 6$ are opposite, and symmetries send opposite corners to opposite corners, so the orbit consists of the colorings whose two white corners are opposite:

$$\mathrm{Orb}_{D_6}(c') = \{\texttt{BBWBBW},\ \texttt{WBBWBB},\ \texttt{BWBBWB}\}.$$

(For instance, $r \circlearrowleft c' =$ `WBBWBB` and $r^2 \circlearrowleft c' =$ `BWBBWB`.)
(C) $|\mathrm{Fix}(g)| = 2^{\mathrm{cyc}(g)}$.

- $r^2 = (1\ 3\ 5)(2\ 4\ 6)$: 2 cycles, so $2^2 = 4$ fixed colorings (the odd corners agree and the even corners agree).
- $r^3 = (1\ 4)(2\ 5)(3\ 6)$: 3 cycles, so $2^3 = 8$.
- $s = (1)(2\ 6)(3\ 5)(4)$: 4 cycles, including the fixed corners 1 and 4, so $2^4 = 16$.
- $rs = (1\ 2)(3\ 6)(4\ 5)$: 3 cycles, so $2^3 = 8$.

(D) A symmetry fixes $c'' = \texttt{BWBWBW}$ exactly when it sends the black corners $\{1,3,5\}$ to themselves. The rotations e, r^2, r^4 move every corner an even number of steps, so they do; r, r^3, r^5 move every corner an odd number of steps and send $\{1,3,5\}$ to $\{2,4,6\}$. Among the reflections, $s = (2\ 6)(3\ 5)$, $r^2s = (1\ 3)(4\ 6)$, and $r^4s = (1\ 5)(2\ 4)$ send $\{1,3,5\}$ to itself, while rs, r^3s, r^5s send corner 1 to the even corners 2, 4, 6. Hence

$$\mathrm{Stab}(c'') = \{e, r^2, r^4, s, r^2s, r^4s\},$$

and by the orbit--stabilizer theorem $|\mathrm{Orb}_{D_6}(c'')| = 12/6 = 2$:

$$\mathrm{Orb}_{D_6}(c'') = \{\texttt{BWBWBW},\ \texttt{WBWBWB}\},$$

where $r \circlearrowright c'' = \texttt{WBWBWB}$.

Orbit–stabilizer in other settings

(A) A rotation that carries a face onto itself keeps the center of that face and the center of the tetrahedron in place, so it turns the tetrahedron about the line through them, which also passes through the opposite vertex. The rotations about this axis are by $0°$, $120°$, and $240°$, so the stabilizer has 3 elements. The orbit of a face consists of all 4 faces, and the tetrahedron has $4 \cdot 3 = 12$ rotations.

(B) The orbit of an edge has 12 elements, so $|\mathrm{Stab}(\textit{edge})| = 24/12 = 2$. The two rotations are the identity and the half turn about the axis through the midpoint of the edge and the midpoint of the opposite edge, which exchanges the two ends of the edge.

(C) Each vertex lies on exactly one of the two triangles. A half turn about a horizontal axis exchanges the two triangles, so it moves every vertex. The other rotations turn the prism about the vertical axis through the centers of the triangles; the rotations by $120°$ and $240°$ move every vertex. So only the identity fixes a vertex, and the prism has $6 \cdot 1 = 6$ rotations: the identity, two turns about the vertical axis, and three half turns.

(D) Symmetries preserve adjacency, so they send sides to sides and diagonals to diagonals; the rotations carry any side to any other side, and any diagonal to the other one. So there are two orbits: the 4 sides $\{1,2\},\{2,3\},\{3,4\},\{1,4\}$, and the 2 diagonals $\{1,3\},\{2,4\}$.

$\mathrm{Stab}(\{1,2\}) = \{e, v\}$: $v = (1\ 2)(3\ 4)$ exchanges 1 and 2, and every other noniden-tity symmetry sends 1 or 2 outside $\{1,2\}$. Check: $4 \cdot 2 = 8$.

$\mathrm{Stab}(\{1,3\}) = \{e, r^2, d_{13}, d_{24}\}$: d_{13} fixes 1 and 3, while r^2 and d_{24} exchange them; r, r^3, v, h send the diagonal to $\{2,4\}$. Check: $2 \cdot 4 = 8$.