Midterm 3 Practice Worksheet 1: Groups and Permutations

Combinatorics

Fall 2026

This worksheet is optional practice for Midterm 3. It is not collected or graded. Work on paper, as you would on the exam, and write the method that justifies each answer. Then compare with the solutions.

This worksheet covers groups, subgroups, cyclic groups, and permutations: the true-or-false facts about groups and the computations with permutations and with $\mathbb{Z}_n$ that appear on the exam. Permutations are composed from right to left: gh means "first h, then g." Cycle types list the cycle lengths, including fixed points, in decreasing order, and $\mathrm{cyc}(g)$ is the number of cycles of g, including fixed points.

1. (True or false)

Decide whether each statement is always true, and give a one-line reason or a counterexample. On the exam, true-or-false questions need no justification, but finding the reason is the best way to check your answer.

(A) The group $\mathbb{Z}_7$, with addition modulo 7, is cyclic.

(B) The symmetric group S_3 is cyclic.

(C) The set $\{0, 3\}$ is a subgroup of $\mathbb{Z}_6$.

(D) In S_4, $(1\ 2)(3\ 4) = (3\ 4)(1\ 2)$.

(E) A group with 15 elements can have a subgroup with 6 elements.

(F) If an element g of a group has order 6, then $g^9 = e$.

(G) The permutation $(1\ 2)(3\ 4\ 5\ 6)$ in S_6 has order 4.

2. (Groups and subgroups)

(A) In $\mathbb{Z}_8$, with addition modulo 8, list the elements of $\langle 2\rangle$, $\langle 4\rangle$, and $\langle 6\rangle$. Is $\mathbb{Z}_8$ cyclic? List all of its generators.

(B) Decide which of the following subsets are subgroups. For each subset that is not a subgroup, give a sum that leaves the subset.

$$\{0, 3, 6\} \subseteq \mathbb{Z}_9, \qquad \{0, 4\} \subseteq \mathbb{Z}_8, \qquad \{0, 1, 7\} \subseteq \mathbb{Z}_8, \qquad \{0, 4, 8\} \subseteq \mathbb{Z}_{12}, \qquad \{0, 5, 10\} \subseteq \mathbb{Z}_{12}.$$

(C) In S_3, compute $(1\ 3)(2\ 3)$ and $(2\ 3)(1\ 3)$. Is S_3 commutative? Find the order of each of the 6 elements of S_3, and decide whether S_3 is cyclic.

(D) A group G has 18 elements. Which of the numbers 4, 6, 9, 12 can be the number of elements of a subgroup of G? Can G contain an element of order 4? An element of order 9?

(E) An element g of a group has order 6. Which of g^{12}, g^{15}, g^{30}, g^{-6} are equal to e? Write g^{40} and g^{-1} in the form g^t with $0 \le t \le 5$.

(F) Find the order of each permutation: (1 2 3 4)(5 6), (1 2)(3 4 5), (1 2 3)(4 5 6), and (1 2 3)(3 4).

Hint. A finite subset of a group is a subgroup exactly when it is closed under the operation. By Lagrange's theorem, the number of elements of a subgroup, and the order of an element, divide the number of elements of the group. The order of a permutation written with *disjoint* cycles is the least common multiple of the cycle lengths; when cycles overlap, compute the product first.

3. (Working with a permutation)

Let g and h be the permutations in S_8 given by

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 4 & 7 & 2 & 1 & 3 & 5 & 8 \end{pmatrix}, \qquad h = (1\ 2\ 4).$$

(A) Write g in cycle notation. Give the cycle type of g and $\mathrm{cyc}(g)$.

(B) Find the order of g and write g^{-1} in cycle notation.

(C) Compute the composition gh (first h, then g) and write it in cycle notation. Give its cycle type.

(D) Compute hg and compare it with gh.

(E) The permutation $q = (1\ 7)(2\ 5\ 9\ 4)$ in S_9 is written with its fixed points omitted. Give the cycle type of q, $\mathrm{cyc}(q)$, and the order of q.

4. (Another permutation)

Let g and h be the permutations in S_8 given by

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 1 & 7 & 4 & 5 & 2 \end{pmatrix}, \qquad h = (2\ 3\ 5).$$

(A) Write g in cycle notation. Give the cycle type of g and $\mathrm{cyc}(g)$.

(B) Find the order of g and write g^{-1} in cycle notation.

(C) Compute the composition gh (first h, then g) and write it in cycle notation.

5. (Cyclic subgroups of $\mathbb{Z}_n$)

(A) In $\mathbb{Z}_{12}$, with addition modulo 12, list the elements of the cyclic subgroups $\langle 8 \rangle$ and $\langle 9 \rangle$, and give the orders of the elements 8 and 9.

(B) Is 5 a generator of $\mathbb{Z}_{12}$? Is 10? Explain, and list all generators of $\mathbb{Z}_{12}$.

(C) In $\mathbb{Z}_9$, list the elements of $\langle 6 \rangle$. Is 6 a generator of $\mathbb{Z}_9$? List all generators of $\mathbb{Z}_9$.

(D) In $\mathbb{Z}_{15}$, list the elements of the cyclic subgroup $\langle 10 \rangle$ and give the order of the element 10. Is 4 a generator of $\mathbb{Z}_{15}$? Explain.

Hint. In $\mathbb{Z}_n$ the "powers" of a are its multiples $a, a+a, a+a+a, \ldots$ reduced modulo n. Keep adding a until you reach 0; the number of elements you listed, including 0, is the order of a.

6. (Symmetries of the regular pentagon)

Label the corners of a regular pentagon $1, 2, 3, 4, 5$ counterclockwise, starting at the bottom, as in the figure. Let r be the rotation by 72° counterclockwise, so $r(i) = i + 1$ (with $5 + 1$ meaning 1), and let s be the reflection across the axis through corner 1 and the center.

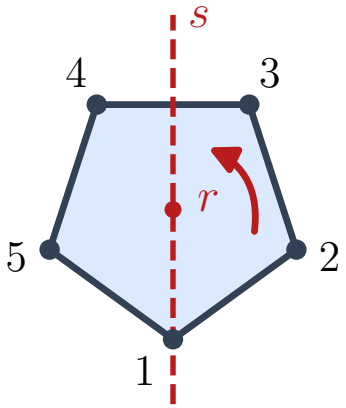


Figure 1: The labeled pentagon, the rotation r, and the mirror line of s.

Figure description: A regular pentagon with corner 1 at the bottom and the other corners counterclockwise: 2 at the lower right, 3 at the upper right, 4 at the upper left, and 5 at the lower left. A dashed vertical mirror line labeled s passes through corner 1 and the center, and a counterclockwise arrow labeled r turns about the center.

(A) Write r, r^2, s, rs, and $r^2 s$ as permutations of the corners in cycle notation.

(B) Compute sr and $r^{-1}s$, and check that they are equal. Is $rs = sr$?

(C) Find the order and cyc of every element of D_5. Is D_5 cyclic?