Homework 8

Combinatorics

Fall 2026

Due: Thursday, November 5, 2026 23:59
Submit: Gradescope, through Blackboard's assignment area.

Please write your homework by hand. Start each exercise on a new page, scan your work to a legible PDF, and clearly mark the pages associated with every exercise. This assignment comes with tables and blank pictures for you to fill in. You may print them and write on them, or copy them by hand.

All four exercises are about one solid, so they build on each other: the table you complete in Exercise 1 is the main tool for the other three.

The pentagonal bipyramid

Glue two pentagonal pyramids together along their pentagonal bases. If every face is an equilateral triangle, the result is the solid shown below, called the **pentagonal bipyramid**. It has 7 vertices and 10 faces.

- The **top vertex** is N and the **bottom vertex** is S.
- The five vertices around the middle, on the **equator**, are $1, 2, 3, 4, 5$, in counterclockwise order when seen from above.
- The upper face with vertices $N, j, j+1$ is called U_j, and the lower face with vertices $S, j, j+1$ is called L_j. Here $5+1$ means 1, so U_5 has vertices $N, 5, 1$.

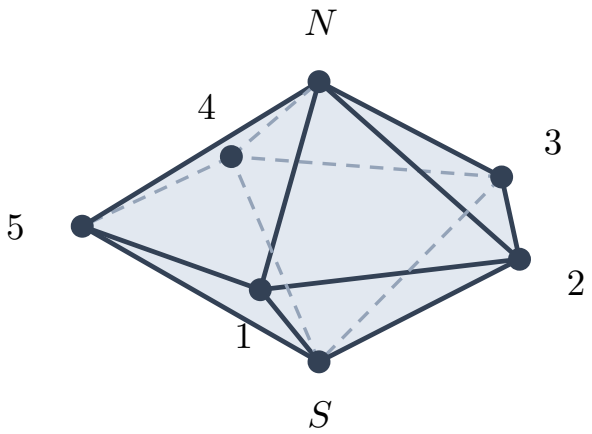


Figure 1: The pentagonal bipyramid. Dashed edges are hidden behind the solid.

Figure description: A pentagonal bipyramid with top vertex N, bottom vertex S, and equatorial vertices 1 to 5 counterclockwise seen from above, vertex 1 in front.

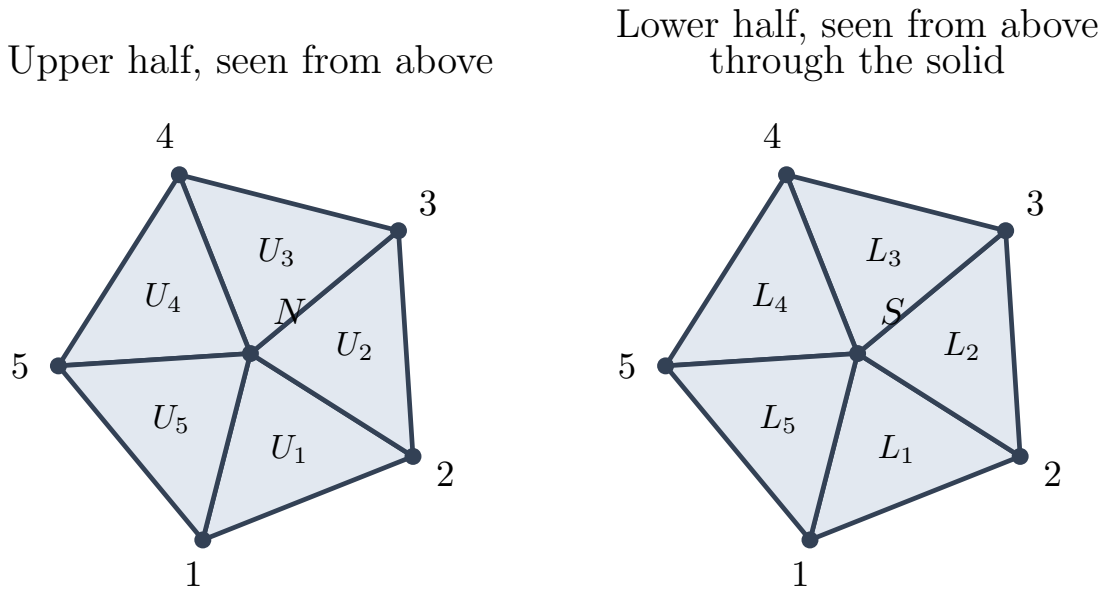


Figure 2: The faces. Both halves are drawn as seen from above, so every equatorial vertex is in the same place in both pictures. On the right, imagine looking down through the solid at the lower faces.

Figure description: Two pentagons divided into five triangles each. Left: upper faces U1 to U5 around N, with U sub j between vertices j and j plus 1. Right: lower faces L1 to L5 around S, in the same positions.

Let G be the set of **rigid motions** of space that carry the solid onto itself. (These are the rigid motions that preserve orientation: you can carry them out by picking up a physical model and putting it back in the same spot. Reflections are *not* allowed.) Composition makes G a group, and G acts on the set of vertices and on the set of faces.

Naming rigid motions. We will see in Exercise 1 that a rigid motion g is completely determined by two pieces of information: where the top vertex N goes and where the equatorial vertex 1 goes. So we name g by the pair

$$[g(N),\, g(1)].$$

For example, $[N, 1]$ is the identity, and $[N, 2]$ is the rigid motion that keeps N on top and moves vertex 1 to where vertex 2 was.

Moving colorings. As in the lecture notes, a rigid motion g moves a coloring c by carrying the color at each vertex x to the vertex $g(x)$. So $g \cdot c$ is the coloring with $(g \cdot c)(g(x)) = c(x)$. The same rule applies to colorings of the faces.

1. (How many rigid motions are there?)

Points: 10

(A) **2 points.** How many edges meet at N? At S? At each equatorial vertex? A rigid motion sends each vertex to a vertex with the same number of edges. Use this to explain why every rigid motion g has $g(N) = N$ or $g(N) = S$.

(B) **2 points.** Take as known that a rigid motion is determined by the two vertices $g(N)$ and $g(1)$. (Once you know where N goes and where 1 goes, the vertex 2 must go to the next vertex around the equator, in the direction fixed by whether the solid is right side up or upside down, and so on.) Explain why there are *at most* 10 rigid motions.

(C) **3 points.** Now show that there are *exactly* 10 rigid motions by showing that every name $[N, j]$ and $[S, j]$ is a real rigid motion.

- For $[N, j]$: describe a rigid motion that turns the solid about the vertical line through N and S and sends 1 to j. What is its angle?
- For $[S, 1]$: take the line through vertex 1 and the midpoint of the edge from 3 to 4, and turn the solid by 180° about it. Explain why this swaps N and S, keeps 1 in place, and carries the solid onto itself.
- For $[S, j]$: explain why doing first $[S, 1]$ and then $[N, j]$ is a rigid motion that sends N to S and 1 to j.

(D) **3 points.** Complete the following table, which records where each rigid motion sends each vertex. Three rows are started for you. Then check your count with the orbit–stabilizer theorem: what is the orbit of the vertex N, and which rigid motions are in $\mathrm{Stab}(N)$?

Rigid motion	$g(N)$	$g(1)$	$g(2)$	$g(3)$	$g(4)$	$g(5)$	$g(S)$
$[N, 1]$	N	1	2	3	4	5	S
$[N, 2]$	N	2	3	4	5	1	S
$[N, 3]$							
$[N, 4]$							
$[N, 5]$							
$[S, 1]$	S	1	5				
$[S, 2]$							
$[S, 3]$							
$[S, 4]$							
$[S, 5]$							

Figure 3: Where each rigid motion sends each vertex.

Figure description: A table with one row per rigid motion [N,1] through [S,5] and columns g(N), g(1), ..., g(5), g(S). Row [N,1] reads N,1,2,3,4,5,S; row [N,2] reads N,2,3,4,5,1,S; row [S,1] starts S,1,5; all other cells are blank.

Need a free hint?. For the rows $[N, j]$, the solid stays right side up, so the equatorial vertices move counterclockwise by the same number of steps as $1 \mapsto j$. For the rows $[S, j]$, the solid is turned upside down. Seen from above, this *reverses* the order around the equator: if $g(1) = j$, then $g(2) = j - 1$, $g(3) = j - 2$, and so on, always counting modulo 5. For example, $[S, 1]$ sends $2 \mapsto 5$, $3 \mapsto 4$, $4 \mapsto 3$, $5 \mapsto 2$.

2. (An orbit and a stabilizer)

Points: 10

We use three colors: red (R), blue (B), and green (G).

- The **vertex coloring** c colors N and S red, 1 and 2 blue, and $3, 4, 5$ green.
- The **face coloring** d colors U_1 and L_5 red, U_3 and L_3 blue, and the other six faces green.

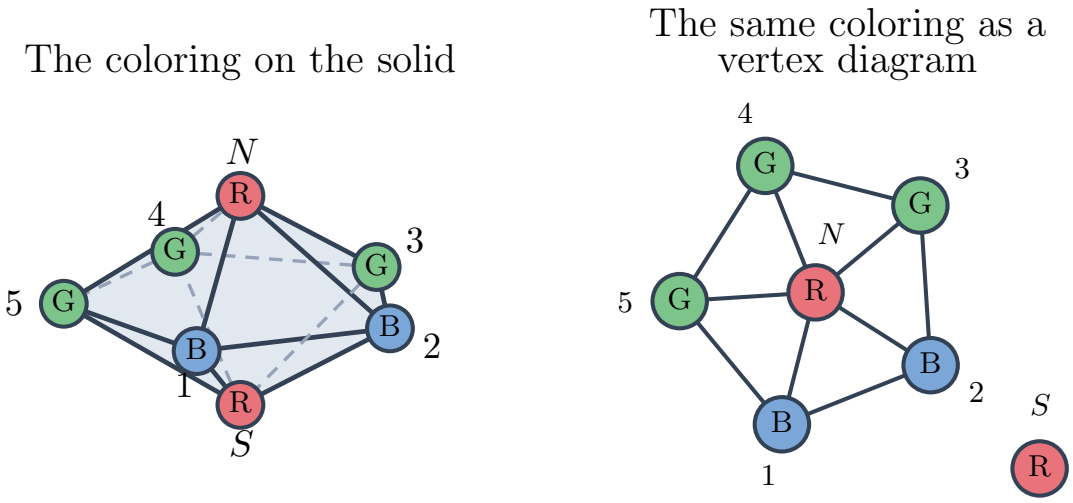


Figure 4: The vertex coloring c. In the vertex diagram on the right, the solid is seen from above: N is in the middle, and S, which is directly below N, is drawn off to the side.

Figure description: The coloring c on the solid and as a flat vertex diagram: N and S red, vertices 1 and 2 blue, vertices 3, 4, 5 green.

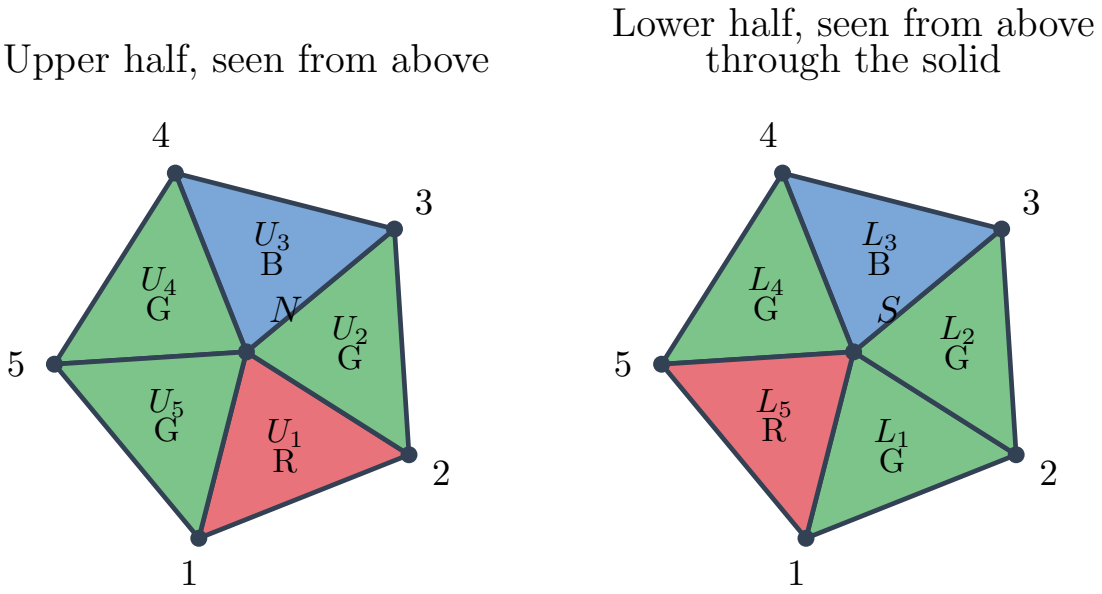


Figure 5: The face coloring d.

Figure description: Upper half: U1 red, U3 blue, U2, U4, U5 green. Lower half: L5 red, L3 blue, L1, L2, L4 green.

(A) **5 points.** *The orbit of c.*

(i) In the blank diagrams at the end of this exercise, draw $g \cdot c$ for each of the 10 rigid motions g, by writing a color letter in every circle. Use your table from Exercise 1: the color at x moves to $g(x)$. The diagram for $[N, 1]$ is done for you.

(ii) How many *different* colorings appear among your 10 diagrams? These colorings form the orbit $G \cdot c$. Describe the orbit in words.

(iii) Which rigid motions g satisfy $g \cdot c = c$? These form the stabilizer $\mathrm{Stab}(c)$. Check that $|G \cdot c| \cdot |\mathrm{Stab}(c)| = 10$.

(B) **5 points.** *The stabilizer of d.* Find $\mathrm{Stab}(d)$ by following these steps.

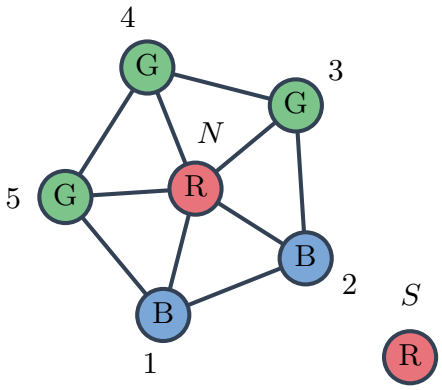
(i) If $g \cdot d = d$, then g must send the red face U_1 to a red face. Explain why this means $g(U_1) = U_1$ or $g(U_1) = L_5$.

(ii) The face $g(U_1)$ is the face whose vertices are $g(N)$, $g(1)$, and $g(2)$. Use your table from Exercise 1 to find every rigid motion with $g(U_1) = U_1$ and every rigid motion with $g(U_1) = L_5$. (You should find exactly one of each.)

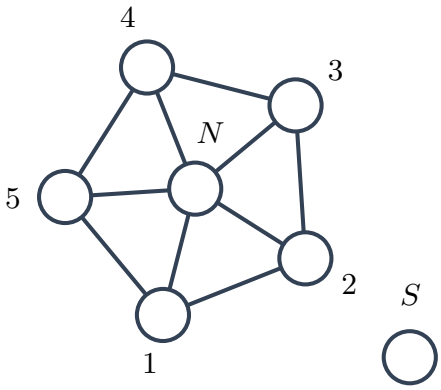
(iii) For each rigid motion you found in (ii), check whether it also sends the blue faces to blue faces and the green faces to green faces. Write down $\mathrm{Stab}(d)$ using the names $[g(N), g(1)]$, and describe each of its elements geometrically.

(iv) Use the orbit–stabilizer theorem to find how many different face colorings are in the orbit $G \cdot d$, without listing them.

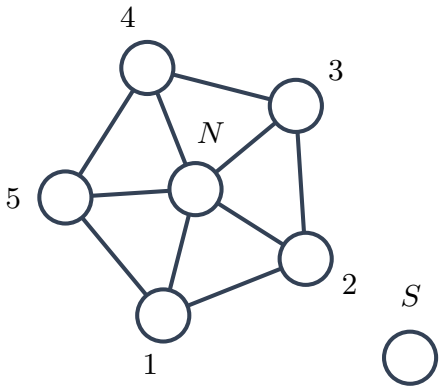
$[N, 1]$



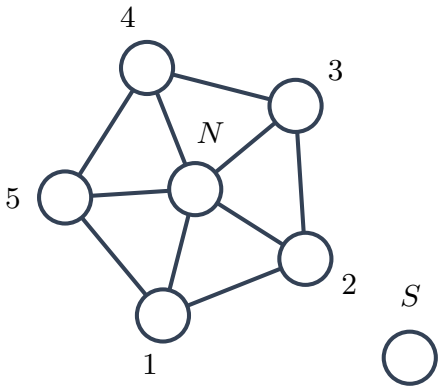
$[S, 1]$



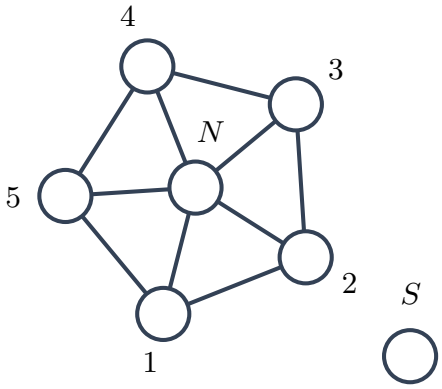
$[N, 2]$



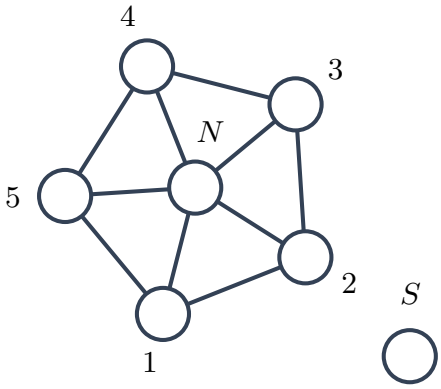
$[S, 2]$



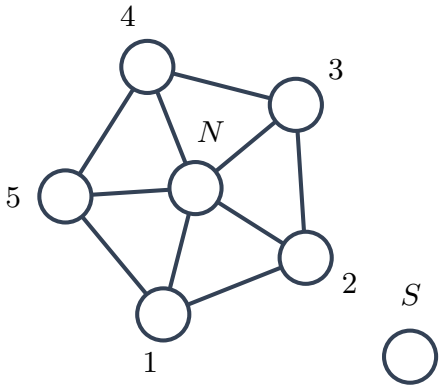
$[N, 3]$



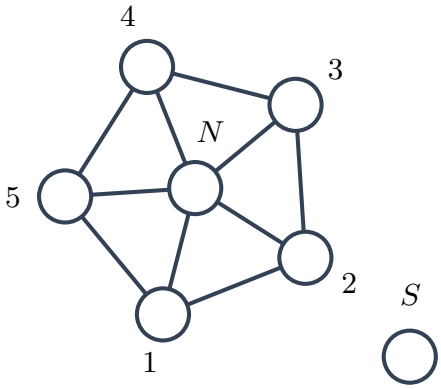
$[S, 3]$



$[N, 4]$



$[S, 4]$



$[N, 5]$



$[S, 5]$



3. (Classifying the rigid motions)

Points: 10

In this exercise you sort the 10 rigid motions into types and record how each one permutes the vertices and the faces.

(A) **3 points.** On the blank solids below, draw the axis of each of the rigid motions $[N, 2]$, $[S, 1]$, $[S, 2]$, and $[S, 3]$, and give the angle of each rigid motion. For the half-turns, say which vertex lies on the axis and which edge the axis passes through the midpoint of.

(B) **4 points.** Complete the table of cycles on vertices. In the column "Axis and angle" write, for example, "vertical, 72°" or "through 1 and the midpoint of 3–4, 180°." In the column "Cycles on vertices" write the permutation $x \mapsto g(x)$ of the 7 vertices in cycle notation, *including* cycles of length 1, and in the last column write the number of cycles.

(C) **3 points.** Complete the table of cycles on faces: the permutation of the 10 faces in cycle notation, including cycles of length 1, and the number of cycles. Then answer: how many rigid motions are there of each of the following three types, and do all rigid motions of the same type have the same numbers of cycles?

- the **identity**;
- **vertical turns**: turns about the vertical axis through N and S by a nonzero angle;
- **half-turns** about an axis through an equatorial vertex.

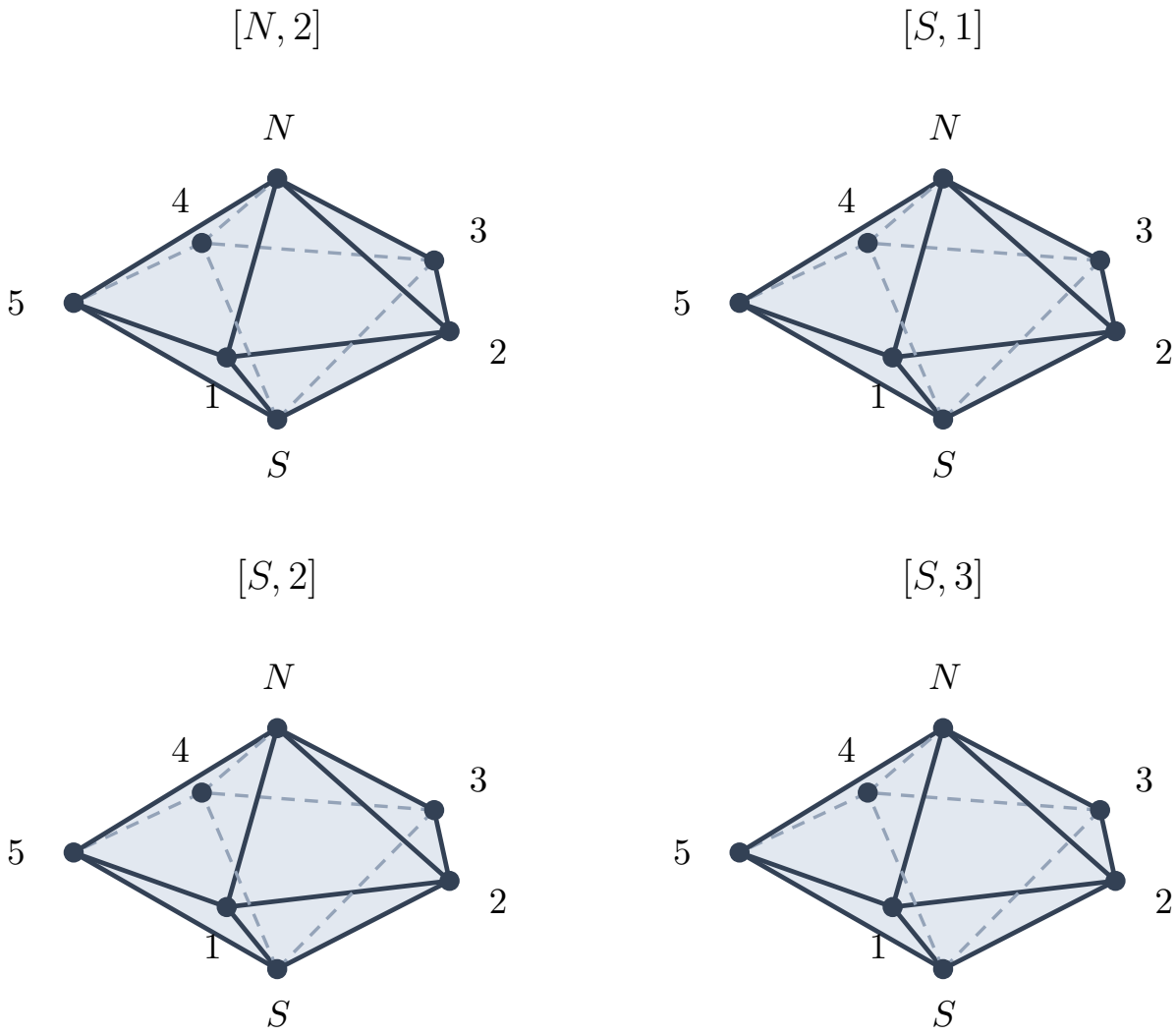


Figure 7: Draw the axis of each rigid motion.

Figure description: Four blank copies of the labeled bipyramid, titled [N,2], [S,1], [S,2], and [S,3].

Rigid motion	Axis and angle	Cycles on vertices	# cycles
$[N, 1]$	identity	$(N)(1)(2)(3)(4)(5)(S)$	7
$[N, 2]$			
$[N, 3]$			
$[N, 4]$			
$[N, 5]$			
$[S, 1]$			
$[S, 2]$			
$[S, 3]$			
$[S, 4]$			
$[S, 5]$			

Figure 8: Cycles on vertices, for Exercise 3(B).

Figure description: A table with one row per rigid motion [N,1] through [S,5] and columns Axis and angle, Cycles on vertices, and number of cycles. Only the identity row is filled in.

Rigid motion	Cycles on faces	# cycles
$[N, 1]$	$(U_1)(U_2)(U_3)(U_4)(U_5)(L_1)(L_2)(L_3)(L_4)(L_5)$	10
$[N, 2]$		
$[N, 3]$		
$[N, 4]$		
$[N, 5]$		
$[S, 1]$		
$[S, 2]$		
$[S, 3]$		
$[S, 4]$		
$[S, 5]$		

Figure 9: Cycles on faces, for Exercise 3(C).

Figure description: A table with one row per rigid motion [N,1] through [S,5] and columns Cycles on faces and number of cycles. Only the identity row is filled in.

Need a free hint?. For the faces, use the same idea as in Exercise 2(B): $g(U_j)$ is the face whose vertices are $g(N), g(j), g(j+1)$, and similarly for $g(L_j)$ with S in place of N. For example, $[S, 1]$ sends U_1, with vertices $N, 1, 2$, to the face with vertices $S, 1, 5$, which is L_5. A face is never fixed by a rigid motion that turns the solid upside down. Why?

4. (Counting colorings up to rigid motion)

Points: 10

Two colorings are **the same up to rigid motion** if some rigid motion carries one to the other, that is, if

they lie in the same orbit. We count the orbits with Burnside's lemma:

$$\text{number of orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|,$$

where $\text{Fix}(g)$ is the set of colorings c with $g \cdot c = c$. Use the three colors R, B, G throughout.

(A) **3 points.** *Vertex colorings fixed by a rigid motion.*

- Explain why a vertex coloring c satisfies $[N, 2] \cdot c = c$ exactly when all five equatorial vertices have the same color. How many such colorings are there? (The colors of N and S are free.)
- Which vertices must have equal colors in a coloring fixed by $[S, 1]$? How many such colorings are there?
- Explain how each answer is $3^{\text{number of vertex cycles}}$, using your table from Exercise 3.

(B) **2 points.** Complete the Burnside table for vertex colorings. Rigid motions of the same type fix the same number of colorings, so each type contributes (how many) $\times$ (fixed colorings). Add the last column, and divide by 10 to find the number of vertex colorings with 3 colors up to rigid motion.

(C) **3 points.** *Face colorings fixed by a rigid motion.* Repeat (A) for colorings of the 10 faces: describe the face colorings fixed by $[N, 2]$ and those fixed by $[S, 1]$ (which faces must have equal colors?), and count them.

(D) **2 points.** Complete the Burnside table for face colorings, and find the number of face colorings with 3 colors up to rigid motion.

Type of rigid motion	How many	Vertex cycles	Fixed colorings (3 colors)	How many $\times$ fixed colorings
identity	1	7	$3^7 = 2187$	2187
vertical turn				
half-turn				
			sum:	
			sum/10:	

Figure 10: Burnside's lemma for vertex colorings, for Exercise 4(B).

Figure description: A table with rows identity, vertical turn, half-turn, sum, and sum divided by 10; columns How many, Vertex cycles, Fixed colorings, and How many times fixed colorings. The identity row reads 1, 7, 3 to the 7th equals 2187, 2187.

Type of rigid motion	How many	Face cycles	Fixed colorings (3 colors)	How many × fixed colorings
identity	1			
vertical turn				
half-turn				
			sum:	
			sum/10:	

Figure 11: Burnside's lemma for face colorings, for Exercise 4(D).

Figure description: The same table for face colorings, with only How many equals 1 filled in for the identity.

Need a free hint?. Check your work: each final answer must be a whole number. As a second check, redo the computation with k colors instead of 3, so that each fixed count becomes $k^{\text{number of cycles}}$. With $k = 1$ color there is exactly one coloring, so your formula must give 1.