Homework 7

Combinatorics

Fall 2026

Due: Thursday, October 29, 2026 23:59
Submit: Gradescope, through Blackboard's assignment area.

Please write your homework by hand. Start each exercise on a new page, scan your work to a legible PDF, and clearly mark the pages associated with every exercise. Compose permutations from right to left, as in the lecture notes: gh means "first h, then g."

1. (Words in D_6)

Points: 10

Recall the presentation

$$D_6 = \langle r, s \mid r^6 = s^2 = e,\ sr = r^{-1}s \rangle$$

of the symmetries of the regular hexagon, with corners labeled $1, \dots, 6$ counterclockwise. Use only the relations, not tables, except where a part asks you to check an answer by computing where corners go.

(A) **2 points.** Prove that $sr^k = r^{-k}s$ for every integer $k \geq 1$ by applying the relations.

(B) **5 points.** Write each of the following words in the normal form $r^j s^\varepsilon$ with $0 \leq j \leq 5$ and $\varepsilon \in \{0, 1\}$:

$$r^4 s r^3 s, \qquad s r^5 s r^2 s r, \qquad (r^2 s)(r^5 s)(r s).$$

Recall that every element of D_6 is either a rotation r^j or one of the six elements $r^j s$, each of which is a reflection of the hexagon across a line through its center. For each word, say whether it is a rotation or a reflection. If it is a rotation, give its angle. If it is a reflection, draw the hexagon with its corners labeled, draw the axis of reflection, and state which corners, if any, lie on the axis. Check your answer by computing where the element sends each corner.

(C) **3 points.** Show that $(r^j s)^2 = e$ for every j. Then find the order of each of the 12 elements of D_6, and check that every order divides $|D_6|$.

2. (Computing in S_9)

Points: 10

Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 9 & 1 & 3 & 8 & 5 & 2 & 6 \end{pmatrix}, \qquad \tau = (1\ 2\ 3)(4\ 5)(6\ 7\ 8\ 9).$$

(A) **2 points.** Write σ in cycle notation, and write τ as a two-row table.

(B) **2 points.** For each of σ and τ, list the lengths of all of its cycles, including cycles of length 1, and state the total number of cycles. Then find the order of σ and the order of τ, that is, the smallest positive integer m with $\sigma^m = e$ (respectively $\tau^m = e$), and explain how you obtain it from the cycle lengths.

(C) **2 points.** Find σ^{-1}, both in cycle notation and as a table.

(D) **2 points.** Compute $\sigma\tau$ and $\tau\sigma$ in cycle notation. Are they equal? List the lengths of the cycles of $\sigma\tau$ and of $\tau\sigma$. Do the two products have the same number of cycles of each length?

(E) **2 points.** Compute σ^{2026} in cycle notation without computing 2026 compositions.

Need a free hint?. For (E), divide 2026 by the order of σ and keep only the remainder. Then handle each cycle of σ separately: a power of a permutation is the composition of the same power of each of its cycles.

3. (The cyclic group $\mathbb{Z}_{12}$)

Points: 10

Recall that in $\mathbb{Z}_n$ the operation is addition modulo n, so the k-th power of a is the remainder of ka on division by n.

(A) **3 points.** Find the order of each of the 12 elements of $\mathbb{Z}_{12}$.

(B) **3 points.** For $a \in \mathbb{Z}_{12}$, the cyclic subgroup $\langle a \rangle$ is the set of all powers of a. In $\mathbb{Z}_{12}$ these are the elements reached by starting from 0 and adding a over and over, always taking the remainder on division by 12. For example, in the smaller group $\mathbb{Z}_6$, adding 2 repeatedly gives $0, 2, 4, 0, 2, 4, ...$, so $\langle 2 \rangle = \{0, 2, 4\}$, and adding 5 repeatedly gives $0, 5, 4, 3, 2, 1, 0, ...$, so $\langle 5 \rangle = \{0, 1, 2, 3, 4, 5\}$.

List the elements of $\langle a \rangle$ for every $a \in \mathbb{Z}_{12}$. How many *different* sets appear in your list? Explain how Lagrange's theorem predicts the possible numbers of elements of these sets.

(C) **2 points.** An element a **generates** $\mathbb{Z}_{12}$ if every element of $\mathbb{Z}_{12}$ is a power of a, that is, if $\langle a \rangle$ is all of $\mathbb{Z}_{12}$: adding a over and over eventually reaches every one of $0, 1, \ldots, 11$. In the example in (B), 5 generates $\mathbb{Z}_6$, but 2 does not.

Use your list from (B) to find all elements that generate $\mathbb{Z}_{12}$. State a rule, in terms of a and 12, that predicts which a are generators.

(D) **2 points.** A group G is **cyclic** if some element $g \in G$ generates it, that is, every element of G is a power of g. For example, $\mathbb{Z}_{12}$ is cyclic, because 1 generates it. Decide whether each of C_6, D_4, and S_3 is cyclic, and justify your answer. Use Lagrange's theorem to explain why D_4 has no subgroup with 3 elements.