

Homework 5 Solutions

Combinatorics

Fall 2026

The write-ups below model the amount of explanation expected in a complete handwritten submission.

0.1 A constrained sum

Problem. Let a_k be the number of ordered triples (u, v, w) of integers satisfying

$$u + v + w = k,$$

where u is nonnegative and even, v is positive and odd, and $10 \leq w \leq 20$.

- (A) Let $U(x)$, $V(x)$, and $W(x)$ be the generating functions for the allowed choices of u , v , and w , respectively. Write each generating function both as a series and in closed form.
- (B) Then find the generating function for the required problem in closed form.
- (C) Use Sage to find the number of solutions for $k = 20$, $k = 35$, and $k = 50$. Include your Sage code and output with your work (screenshot is fine).

Solution.

Approach. Each variable gets its own generating function, in which the exponent records the value of that variable. Choosing u , v , and w independently and adding their values corresponds to multiplying the three generating functions.

(A) The allowed values of u are $0, 2, 4, \dots$, so

$$U(x) = 1 + x^2 + x^4 + x^6 + \dots = \frac{1}{1 - x^2}.$$

The allowed values of v are $1, 3, 5, \dots$, so

$$V(x) = x + x^3 + x^5 + \dots = x(1 + x^2 + x^4 + \dots) = \frac{x}{1 - x^2}.$$

The allowed values of w are $10, 11, \dots, 20$, so

$$W(x) = x^{10} + x^{11} + \dots + x^{20} = x^{10}(1 + x + \dots + x^{10}) = \frac{x^{10}(1 - x^{11})}{1 - x}.$$

(B) A solution of $u + v + w = k$ corresponds to one term x^u from $U(x)$, one term x^v from $V(x)$, and one term x^w from $W(x)$ whose product is x^k . Therefore

$$A(x) = \sum_{k \geq 0} a_k x^k = U(x)V(x)W(x) = \frac{x^{11}(1 - x^{11})}{(1 - x)(1 - x^2)^2}.$$

(C) The answers are the coefficients $[x^{20}]A(x)$, $[x^{35}]A(x)$, and $[x^{50}]A(x)$. The series must go past x^{50} .

```
var('x')
A = x^11 * (1 - x^11) / ((1 - x) * (1 - x^2)^2)
S = A.series(x, 51)
print(S.coefficient(x, 20), S.coefficient(x, 35), S.coefficient(x, 50))
```

Output:

15 63 90

Therefore

$$a_{20} = 15, \quad a_{35} = 63, \quad a_{50} = 90.$$

Check for $k = 20$. Since $u + v$ is odd, w must be odd, so $w \in \{11, 13, 15, 17, 19\}$. Then $u + v = 9, 7, 5, 3, 1$, and v can be any odd number from 1 to $u + v$. This gives $5 + 4 + 3 + 2 + 1 = 15$ solutions.

0.2 Computing a closed formula for a sum of cubes

Problem. In this exercise we use generating functions to compute a closed formula for the sum of the cubes of numbers j from 0 to k . Follow the steps below and show the computations used in each step.

- (A) Write down the generating function in closed form for the sequence $1, 1, 1, 1, 1, 1, \dots$
- (B) Find the generating function in closed form for the sequence $0, 1, 2, 3, 4, 5, 6, \dots$
- (C) Find the generating function in closed form for the sequence $0, 1 \cdot 1 \cdot 1, 2 \cdot 2 \cdot 2, \dots$
- (D) Find the generating function in closed form for the sum of cubes up to k .
- (E) Find

$$[x^m] \frac{1}{(1-x)^5}$$

and use that to extract the coefficient of x^k from $Q(x)$.

Solution.

Approach. Applying the operator $x \frac{d}{dx}$ to $\sum_j c_j x^j$ gives $\sum_j j c_j x^j$, so it multiplies the j th coefficient by j . Multiplying by $\frac{1}{1-x}$ replaces a sequence by its sequence of partial sums.

(A) By the geometric series,

$$\sum_{j \geq 0} x^j = \frac{1}{1-x}.$$

(B) Apply $x \frac{d}{dx}$ to part (A):

$$\sum_{j \geq 0} j x^j = x \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{x}{(1-x)^2}.$$

(C) Apply $x \frac{d}{dx}$ twice more. First,

$$\sum_{j \geq 0} j^2 x^j = x \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right) = x \cdot \frac{(1-x) + 2x}{(1-x)^3} = \frac{x(1+x)}{(1-x)^3}.$$

Then

$$\begin{aligned} \sum_{j \geq 0} j^3 x^j &= x \frac{d}{dx} \left(\frac{x+x^2}{(1-x)^3} \right) = x \cdot \frac{(1+2x)(1-x) + 3(x+x^2)}{(1-x)^4} \\ &= x \cdot \frac{1+x-2x^2+3x+3x^2}{(1-x)^4} = \boxed{\frac{x(1+4x+x^2)}{(1-x)^4}}. \end{aligned}$$

(D) Let $q_k = 0^3 + 1^3 + \dots + k^3$ and $Q(x) = \sum_{k \geq 0} q_k x^k$. Since

$$\frac{1}{1-x} \sum_{j \geq 0} c_j x^j = \sum_{k \geq 0} \left(\sum_{j=0}^k c_j \right) x^k,$$

multiplying part (C) by $\frac{1}{1-x}$ gives

$$Q(x) = \boxed{\frac{x(1+4x+x^2)}{(1-x)^5} = \frac{x+4x^2+x^3}{(1-x)^5}}.$$

(E) The coefficient of x^m in $(1-x)^{-5}$ counts nonnegative solutions of $y_1 + \dots + y_5 = m$, so by stars and bars

$$[x^m] \frac{1}{(1-x)^5} = \boxed{C(m+4, 4)}.$$

Each term x^s in the numerator of $Q(x)$ shifts exponents by s , so

$$q_k = [x^k] Q(x) = C(k+3, 4) + 4C(k+2, 4) + C(k+1, 4).$$

Here $C(n, 4) = \frac{n(n-1)(n-2)(n-3)}{24}$, which is 0 for $n = 0, 1, 2, 3$, so this formula is valid for all $k \geq 0$. Factor out $\frac{(k+1)k}{24}$:

$$\begin{aligned} q_k &= \frac{(k+1)k}{24} \left[(k+3)(k+2) + 4(k+2)(k-1) + (k-1)(k-2) \right] \\ &= \frac{(k+1)k}{24} \left[(k^2 + 5k + 6) + (4k^2 + 4k - 8) + (k^2 - 3k + 2) \right] \\ &= \frac{(k+1)k}{24} \cdot 6k(k+1). \end{aligned}$$

Therefore

$$\boxed{0^3 + 1^3 + \dots + k^3 = \frac{k^2(k+1)^2}{4} = \left(\frac{k(k+1)}{2} \right)^2}.$$

Check: for $k = 3$, $1 + 8 + 27 = 36 = 6^2$.

0.3 Compositions with parts 1, 2, and 3

Problem. Let r_k be the number of ordered compositions of k whose parts all belong to $\{1, 2, 3\}$. Include one empty composition when $k = 0$. For example, $(1, 3)$ and $(3, 1)$ are different compositions of 4.

- (A) Write the generating function $M(x)$ for one part, and explain why $M(x)^m$ counts compositions having exactly m parts.
- (B) Sum over all possible numbers of parts to prove that

$$R(x) = \sum_{k \geq 0} r_k x^k = \sum_{m \geq 0} M(x)^m = \frac{1}{1 - x - x^2 - x^3}.$$

- (C) Use Sage to find r_{30} . Also report the list $(r_0, r_1, \dots, r_{12})$. Include your Sage code and output with your work.

Solution.

(A) One part has value 1, 2, or 3, and the exponent records its value:

$$M(x) = x + x^2 + x^3.$$

Expanding the product

$$M(x)^m = \underbrace{(x + x^2 + x^3) \cdots (x + x^2 + x^3)}_{m \text{ factors}}$$

means choosing one term x^{c_i} from the i th factor for each $i = 1, \dots, m$. This choice is exactly the ordered list $(c_1, \dots, c_m)$, and it contributes $x^{c_1 + \dots + c_m}$. So $[x^k]M(x)^m$ is the number of compositions of k with exactly m parts from $\{1, 2, 3\}$. Different orders come from different choices of terms, so they are counted separately. For $m = 0$, $M(x)^0 = 1$ counts the empty composition of 0.

(B) Every composition has exactly one number of parts $m \geq 0$. Summing over m therefore counts every composition exactly once:

$$R(x) = \sum_{m \geq 0} M(x)^m.$$

Since $M(x)$ has no constant term, $M(x)^m$ only contains powers x^k with $k \geq m$. Thus each coefficient of the sum is a finite sum, and the geometric-series identity applies:

$$R(x) = \frac{1}{1 - M(x)} = \boxed{\frac{1}{1 - x - x^2 - x^3}}.$$

(C)

```
var('x')
R = 1 / (1 - x - x^2 - x^3)
T = R.taylor(x, 0, 30)
print(T.coefficient(x, 30))
print([T.coefficient(x, k) for k in range(13)])
```

Output:

53798080

[1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927]

Therefore

$$r_{30} = 53,798,080$$

and

$$(r_0, \dots, r_{12}) = (1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927).$$

Check: the compositions of 3 are (3), (1, 2), (2, 1), (1, 1, 1), so $r_3 = 4$.

0.4 Ordered compositions versus restricted partitions

Problem. Let p_k be the number of ways to write k as a sum of parts chosen from

$$\{1, 3, 4, 6\},$$

where each part size may be used any number of times and order is irrelevant. Thus $1 + 3 + 3$ and $3 + 1 + 3$ represent the same choice of parts.

- (A) Model $(p_k)_{k \geq 0}$ with an ordinary generating function $P(x)$. Write and explain one factor for each allowed part size. In particular, explain why this model is a product of factors rather than the sequence construction used in Exercise 3.
- (B) Use Sage to compute p_{25} , p_{50} , and p_{100} . Include your Sage code and output with your work. Make sure your Taylor expansion includes the coefficient of x^{100} .

Solution.

(A) Since order is irrelevant, a choice of parts is determined by the numbers $n_1, n_3, n_4, n_6 \geq 0$ of parts of each size. Its total is

$$1 \cdot n_1 + 3n_3 + 4n_4 + 6n_6 = k.$$

The parts of size s contribute $0, s, 2s, 3s, \dots$ to the total, so the factor for size s is

$$1 + x^s + x^{2s} + x^{3s} + \dots = \frac{1}{1 - x^s}.$$

Choosing one term from each factor is choosing (n_1, n_3, n_4, n_6) , so

$$P(x) = \frac{1}{(1 - x)(1 - x^3)(1 - x^4)(1 - x^6)}.$$

In Exercise 3, the factors of $M(x)^m$ correspond to the first part, second part, and so on, so (1,3) and (3,1) come from different choices of terms and are counted twice. Here each factor corresponds to a part size, not a position,

so $1 + 3 + 3$ and $3 + 1 + 3$ both correspond to the single choice $n_1 = 1, n_3 = 2, n_4 = n_6 = 0$ and are counted once.

(B)

```
var('x')
P = 1 / ((1 - x) * (1 - x^3) * (1 - x^4) * (1 - x^6))
T = P.taylor(x, 0, 100)
print(T.coefficient(x, 25), T.coefficient(x, 50), T.coefficient(x, 100))
```

Output:

75 425 2835

Therefore

$$p_{25} = 75, \quad p_{50} = 425, \quad p_{100} = 2835.$$